

Money Demand and Macroeconomic Stability Revisited

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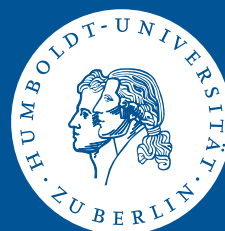
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Money Demand and Macroeconomic Stability Revisited¹

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Abstract

This paper examines how money demand induced real balance effects contribute to the determination of the price level, as suggested by Patinkin (1949,1965), and if they affect conditions for local equilibrium uniqueness and stability. There exists a unique price level sequence that is consistent with an equilibrium under interest rate policy, only if beginning-of-period money enters the utility function. Real money can then serve as a state variable, implying that interest rate setting must be passive for unique, stable, and non-oscillatory equilibrium sequences. When end-of-period money provides utility, an equilibrium is consistent with infinitely many price level sequences, and equilibrium uniqueness requires an active interest rate setting. The stability results are, in general, independent of the magnitude of real balance effects, and apply also when prices are sticky. In contrast, under a constant money growth policy, equilibrium sequences are (likely to be) locally stable and unique for all model variants.

JEL classification: E32, E41, E52.

Keywords: Real balance effects, predetermined money, price level determination, real determinacy, monetary policy rules.

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Non-technical summary

Money demand does not play a prominent role in recent studies on monetary policy. Typically, monetary aggregates considered in a way that makes the demand for money de facto irrelevant for the determination of macroeconomic aggregates when the central bank realizes certain interest rate targets. This has even led to the theoretical concept of economies being cashless in the limit, which facilitates an analysis of interest rate policy where monetary aggregates can be neglected.

In this paper, we examine whether money demand is indeed negligible for the short-run behavior of macroeconomic aggregates and for monetary policy. The main conclusion is that money demand matters for the design of a stabilizing monetary policy if the outstanding stock of money effectively restricts households' consumption decisions. The price level can then be non-neutral with regard to real activity and the inflation rate, that is, the classical dichotomy fails and purely nominal changes have real effects. In this case, a central bank should only moderately adjust its interest rate target in response to changes in the state of the economy. The reason is that interest rate changes do not only affect households' savings and consumption expenditures due to their willingness to smooth their consumption stream, but also due to changes in money holdings that become more costly under high interest rates. In contrast, the current consensus view on stabilizing interest rate policy disregards the latter effect and calls for interest rates to be adjusted highly reactive (actively) to changes in inflation.

A consequence of money holdings being essential for households' goods market transactions is that strong interest rate adjustments can destabilize the economy, i.e., can lead to unstable output dynamics and to hyperinflations. Such destabilizing effects can be triggered off by fundamental disturbances, and do therefore not rely on origins of fluctuations that are speculative in nature, as suggested in various related studies (emphasizing the danger of inappropriate policy responses to shifts in expectations). Given that the particular relation between money holdings and household transactions is decisive for the impact of monetary policy, there is no general principle for the design of interest rate targets that stabilize the economy under different specifications of money demand. Instead, the only robust stabilizing device to abandon unstable macroeconomic outcomes is a regime switch, from an interest rate policy to a regime that controls the supply of money by holding the growth rate of nominal balances constant. In this case, monetary policy has a non-destabilizing impact on the economy regardless whether the stock of money is essential for current consumption or not.

The analysis is based on a dynamic general equilibrium model, where money enters a non-separable utility function and prices are either perfectly or imperfectly flexible. Then, households' behavior is influenced by real balance effects, which typically arise when transaction frictions are modeled in explicit way, for example in form of shopping time or real resource costs. We apply two different assumptions about the particular stock of money that enters the utility function, either the stock of money held at the beginning or at the end of the period. The former assumption corresponds the idea that the goods market opens before the asset market. The second assumption can be interpreted as a shortcut for a specification where households can always adjust their money holdings in accordance with their current

transactions. Thus, both versions substantially differ with regard to the role of money held by households at the beginning of each period. In the former version, beginning-of-period real balances restrict households' current consumption expenditures. In the latter version, households adjust their end-of-period money holdings in accordance with their current consumption expenditures. Hence, beginning-of-period real balances are then determined by the previous period consumption decision such that the causality is reversed.

Throughout the paper we focus on the role of money demand and monetary policy rules for the determination of locally stable equilibrium sequences at the steady state. We show that the existence (not the magnitude) of real balance effects contribute to price level determinacy, as for example suggested by Patinkin (1965), if the stock of money held at the beginning of the period rather than held at the end of the period is assumed to provide transaction services. Then, there exists a unique initial price level that is consistent with a rational expectations equilibrium. Whenever the price level can uniquely be determined, real money serves as a relevant state variable, since money that has been acquired in the previous period relates to households' current consumption expenditures.

These properties, which have until now been ignored in the literature, crucially affect the conditions that ensure macroeconomic stability under interest rate feedback rules when prices are flexible or sticky. Then, the nominal interest rate should be adjusted by less than one for one with inflation (passively) to ensure unique and locally stable equilibrium sequences. If, however, consumption relates to the end-of-period stock of money, then the equilibrium displays price level indeterminacy. In this case, the conditions for uniqueness and stability of equilibrium sequences correspond to the so-called "Taylor-principle" that applies to cashless economies. While the stability conditions for an interest rate policy regime are highly sensitive, local stability and uniqueness of equilibrium sequences are likely to be ensured under a policy regime that keeps the growth rate of nominal balances constant. This result therefore indicates that under non-negligible transaction frictions the central bank should rather control the supply of money than the nominal interest rate to avoid unstable macroeconomic dynamics. Nonetheless, an optimal conduct of monetary policy will certainly require the supply of money to respond to the state of the economy. We plan to investigate such contingent money supply policies in future work.

1 Introduction

The conduct of monetary policy is known to affect the determination of the price level and, under non-neutrality, the real equilibrium allocation. Recent studies to this line of research, mainly focus on policy regimes summarized by interest rate feedback rules, such as Benhabib et al. (2001a, 2001b, 2003) or Carlstrom and Fuerst (2001),⁴ while previous contributions to this literature have primarily considered monetary policy regimes that are characterized by constant money growth (see, e.g., Obstfeld and Rogoff, 1983, Matsuyama, 1990, 1991, or Matheny, 1998). Correspondingly, researchers nowadays pay less attention to the role of monetary aggregates and increasingly employ money demand specifications that allow to neglect money for the analysis of equilibrium determination (see Dupor, 2001, Woodford, 2003, or Carlstrom and Fuerst, 2004).

In this paper we (re-)assess the role of money for the determination of the price level, and for uniqueness and stability of equilibrium sequences. In particular, we examine the case where the evolution of monetary aggregates is non-negligible due to real balance effects, which typically characterize households' consumption behavior when transaction services of money are modelled in an explicit way.⁵ The classical dichotomy is invalid under this assumption, which might contribute to the determination of the price level under flexible prices, as shown by Patinkin (1949, 1965). We find that uniqueness of the price level under interest rate policy crucially relies on whether the stock of money held at the beginning of the period or at the end of the period is assumed to provide transaction services. In the latter case, a rational expectations equilibrium is consistent with any initial price level and, thus, with multiple price level sequences. If, however, beginning-of-period money relates to households' transactions, then a rational expectations equilibrium is associated with a unique price level sequence (nominal determinacy, as defined by Benhabib et al., 2001). A rational expectations equilibrium can further be uniquely determined (real determinacy), if interest rates are set contingent on current inflation rate. Then, real money becomes a relevant predetermined state variable, which implies that interest rate policy should be passive to avoid explosive or oscillatory equilibrium sequences.⁶ When end-of-period money is assumed to provide transaction services, real balance effects turn out to be (almost) negligible for equilibrium determination, and the principles for real determinacy in cashless economies (see Woodford, 2003) apply.⁷

There are a variety of means to induce demand for money, i.e., a non-interest bearing government liability, in general equilibrium models. Most of them, for example, cash-in-advance constraints (Clower, 1967), real resource costs of transactions (Feenstra, 1986), or shopping time specifications (McCallum and Goodfriend, 1987), refer to the transaction role of money. An alternative way to induce households to hold a positive amount of money

⁴See also De Fiore and Liu (2004), Gali et al. (2004), Meng (2002), or Weder (2004)

⁵Real balance effects can thus be viewed as a reasonable property, since if money is assumed to provide transaction services then these benefits should be related to households' actual volume of transactions (see McCallum, 2001, or Woodford, 2003).

⁶An interest rate peg can, however, not lead to a uniquely determined rational expectations equilibrium and, therefore, leads to multiple price level sequences, which contrasts the result in Benassy (2000), who introduces real balance effects via an overlapping generations structure.

⁷It should be noted that a rational expectations equilibrium is consistent with any initial price level (nominal indeterminacy), if there are no real balance effects.

is to assume that money enters the utility (MIU) function, which originates in Sidrauski's (1967) seminal paper. This is probably the most widely applied approach to money demand in the recent literature on monetary policy analysis (see, e.g., Woodford, 2003), and is often viewed as being closely related to the aforementioned specifications. In fact, Brock (1974) and Feenstra (1986) have shown that assuming MIU can be equivalent to the more explicit specifications of transaction frictions. However, for an exact equivalence, real balances and consumption should enter the utility function in a non-separable way. While this property is commonly neglected, since real balance effects are typically found to be very small (see, e.g., Lucas, 2000, or Ireland, 2003), we show that this can have substantial consequences for equilibrium determination, which under sticky prices does not rely on the magnitude of the real balance effect.

We develop a discrete time dynamic general equilibrium model, where money enters a non-separable utility function and prices are either completely flexible or set in a staggered way. We apply two different assumptions about the particular stock of money that enters the utility function, either the stock of money held at the beginning or at the end of the period. The former assumption corresponds to Svensson's (1985) cash-in-advance specification, where the goods market opens before the asset market, and is, for example, applied in Woodford (1990), McCallum and Nelson (1999), or Lucas (2000). The second assumption can be interpreted as a short-cut for a specification where households can always adjust their money holdings in accordance with their current transactions. It avoids Hahn's (1965) paradox in finite horizon general equilibrium models and is now widely applied in infinite horizon models (see, e.g., Woodford, 2003).

We focus on the role of money demand and monetary policy rules for the determination of locally stable equilibrium sequences at the steady state. It turns out that the unique determination of the price level relates (under completely flexible prices) to the property of real balances to serve as a relevant state variable.⁸ For a rational expectations equilibrium to be characterized by this property, beginning-of-period money has to enter the utility function. On the contrary, real balances never serve as a relevant state variable if end-of-period money provides utility. Thus, both versions substantially differ with regard to the role of money held by households at the beginning of each period. In the former version, beginning-of-period real balances restrict household's current consumption expenditures. In the latter version, households adjust their end-of-period money holdings in accordance with their current consumption expenditures. Hence, beginning-of-period real balances are then determined by the previous period consumption decision such that the causality is reversed. For the beginning-of-period value of real balances to be, actually, relevant for the determination of a rational expectations equilibrium, there must further exist a unique price level sequence consistent with equilibrium. Put differently, unless there is a uniquely determined price level, there are multiple real values for the beginning-of-period stock of money, which are consistent with a rational expectations equilibrium.

Thus, under real balance effects and interest rate policy, a uniquely determined price level is associated with real money being a relevant state variable and, thus, with a his-

⁸This property marks a main difference of our framework to the specifications examined in Carlstrom and Fuerst (2001), who show that different assumptions about the timing of markets can affect the conditions for (real) equilibrium determinacy under interest rate rules.

tory dependent evolution of equilibrium sequences, which crucially affects the conditions for macroeconomic stability. This property has been disregarded in related studies where money demand specifications are applied which relate to our end-of-period version (see Benhabib et al., 2001a, Carlstrom and Fuerst, 2001, or Woodford, 2003). The main principles for equilibrium determination and stability under simple monetary policy feedback rules and *flexible prices* can be summarized as follows.⁹

- For the unique determination of a rational expectations equilibrium and a consistent price level sequence, i.e., for real and nominal determinacy, beginning-of-period money has to enter the utility function, and interest rate policy has to respond to current inflation. Neither an interest rate peg nor a forward looking interest rate rule lead to this result.
- Under the beginning-of-period specification, an interest rate policy that reacts to changes in current inflation has to be passive for equilibrium sequences to be uniquely determined and to converge to the steady state in a non-oscillatory way.
- If the end-of-period specification applies or expected future inflation serves as the policy indicator, the equilibrium displays nominal indeterminacy, and interest rate policy has to be active for uniqueness of equilibrium sequences.
- Under a constant money growth regime, equilibrium sequences are consistent with any initial price level and real money does not serve as a relevant state variable (though, monetary policy is history dependent). Equilibrium sequences are, in any case, locally stable and uniquely determined.¹⁰

Throughout the analysis we take into account that (stable) equilibrium sequences can be non-oscillatory or oscillatory, given that the latter property can hardly be viewed as recommendable for a central bank that aims to stabilize the economy. In the second part of the paper, we examine the role of real balance effects for the case where prices are sticky (set in a staggered way), implying that the price level can always be determined. Under this specification, which has scarcely been examined for related purposes,¹¹ real balances serve as a relevant predetermined state variable for all aforementioned policy rules, when the beginning-of-period specification applies. If, however, the end-of-period stock of money enters the utility function, households are entirely forward looking, and real money serves as a relevant state variable only if monetary policy is history dependent, i.e., when the central bank applies a money growth rule. Nonetheless, the determinacy properties under constant

⁹To be more precise, these results apply for finite labor supply elasticities.

¹⁰This result relies on real balance effects to imply consumption and real balances to be Edgeworth-complements. When they are Edgeworth-substitutes, constant money growth can also lead to real indeterminacy, as shown by Carlstrom and Fuerst (2003).

¹¹Exceptions are Benhabib et al. (2001b) and Kurozumi (2004), providing determinacy conditions for interest rate rules. Benhabib et al. (2001b) apply a continuous time framework, which corresponds to our version where end-of-period money enters the utility function. Kurozumi (2004) examines the determinacy and E-stability implications of Taylor-rules in a discrete time framework, and derives results that are consistent with our findings for the end-of-period specification.

money growth and sticky prices, which have until now not been analytically assessed,¹² are shown to correspond to those under flexible prices. The main implications for equilibrium uniqueness and stability under *sticky prices* are as follows:

- When beginning-of-period money provides utility, interest rate policy has to be passive to lead to locally stable, unique, and non-oscillatory equilibrium sequences, regardless whether current or future inflation enters the policy rule. An active interest rate policy is associated with locally stable and unique equilibrium sequences if and only if end-of-period money provides utility and current inflation serves as the policy indicator.
- As under flexible prices, the central bank can ensure equilibrium sequence to be uniquely determined, locally stable, and non-oscillatory under both timing specifications by holding the growth rate of money constant, provided that real balance effects are not extremely large.

While these results are derived for the case where the labor supply elasticity is finite, we further show that the assumption of an infinitely elastic labor supply, which is for example applied in Dopor (2001), Carlstrom and Fuerst (2004), or Weder (2004) for a related purpose, is not harmless for the local equilibrium properties under interest rate policy. For example, we find that an equilibrium under interest rate policy and flexible prices is then consistent with any initial price level, and that the well-established principles for real determinacy under sticky prices for a separable utility function (see Woodford, 2003) apply when end-of-period money provides utility.

The remainder of the paper is organized as follows. Section 2 develops the model. Section 3 analyses nominal and real determinacy under flexible prices. In the first part, we consider the case where the beginning-of-period stock of money provides utility, while the results for the end-of-period specification are briefly summarized in the second part.¹³ For both specifications, we derive the implications for equilibrium determination and local stability under current and forward looking interest rate rules, and for money growth rules. The last part of section 3 discusses our findings and compares them to results in related studies. The first two parts of section 4, which focuses on the conditions for real determinacy when prices are set in a staggered way, are structured as in previous section. The third part of section 4 discusses the main mechanisms, which are responsible for the results. Section 5 concludes.

2 The model

In this section an infinite horizon general equilibrium model with representative agents is developed. We consider a money in the utility function specification that leads to real balance effects. We further allow for prices to be set in a staggered way to facilitate comparisons with related studies, such as Carlstrom and Fuerst (2001) or Woodford (2003). Monetary policy is either specified in form of an interest rate feedback rule or constant money growth. To check

¹²Evans and Honkapohja (2003) provide a numerical analysis of real determinacy under constant money growth in a sticky price framework without real balance effects.

¹³Our findings for the latter case correspond to the results in Benhabib et al. (2001a), Carlstrom and Fuerst (2001), and Woodford (2003)

for the robustness of the results for the former policy regime, we apply contemporaneous and forward looking interest rate rules. Uncertainty is due to an aggregate productivity shock which realizes at the beginning of the period.

Lower (upper) case letters denote real (nominal) variables. There is a continuum of identical and infinitely lived households. At the beginning of period t , households' financial wealth comprises money M_{t-1} , a portfolio of state contingent claims on other households yielding a (random) payment Z_t , and nominally non-state contingent government bonds B_{t-1} carried over from the previous period. Let $q_{t,t+1}$ denote the period t price of one unit of currency in a particular state of period $t+1$ normalized by the probability of occurrence of that state, conditional on the information available in period t . Then, the price of a random payoff Z_{t+1} in period $t+1$ is given by $E_t[q_{t,t+1}Z_{t+1}]$. The households' budget constraint reads

$$M_t + B_t + E_t[q_{t,t+1}Z_{t+1}] + P_t c_t \leq R_{t-1}B_{t-1} + M_{t-1} + Z_t + P_t w_t l_t + P_t \omega_t - P_t \tau_t, \quad (1)$$

where c_t denotes consumption, P_t the aggregate price level, w_t the real wage rate, l_t working time, τ_t a lump-sum tax, R_t the gross nominal interest rate on government bonds, and ω_t profits of firms. Further, households have to fulfill the no-Ponzi game condition, $\lim_{i \rightarrow \infty} E_t q_{t,t+i}(M_{t+i} + B_{t+i} + Z_{t+i+1}) \geq 0$. The objective of the representative household is

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, A_t/P_t), \quad \beta \in (0, 1), \quad (2)$$

where β denotes the subjective discount factor and A_t nominal balances, which will be defined below. The instantaneous utility function is assumed to satisfy

$$u_c > 0, u_l < 0, u_a > 0, u_{cc} < 0, u_{aa} < 0, u_{ll} \leq 0, \quad (3)$$

$$u_{ca} > 0, u_{cl} = u_{al} = 0, u_{cc}u_{aa} - u_{ca}^2 > 0, \quad (4)$$

and the usual Inada-conditions, where $a_t = A_t/P_t$. According to (4) the cross derivative u_{ca} is (strictly) positive, such that marginal utility of consumption rises with real money balances. The resulting properties, i.e., non-separability and real balance effects, typically emerge under more explicit specifications of transaction frictions. As, for example, shown by Brock (1974) or Feenstra (1986), a money-in-the-utility (MIU) function specification, which is equivalent to a specification where purchases of consumption goods are associated with transaction costs that are either measured by shopping time or real resources, is usually characterized by these properties. To be more precise, introducing these transaction friction in a corresponding model with a utility function $v(c_t, 1 - l_t)$ would lead to real balance effects, which are equivalent to a MIU specification with $u_{ca} > 0$, if (but not only if) the labor supply elasticity is finite (see appendix 6.1). It should be noted that an infinite labor supply elasticity will lead to be of particular interest in what follows.

To avoid additional complexities, we assume that the respective cross derivatives are equal to zero $u_{lc} = u_{la} = 0$. The last assumption in (4), $u_{cc}u_{aa} - u_{ca}^2 > 0$, is imposed to ensure – together with $u_{cc} < 0$ and $u_{aa} < 0$ (see 3) – the utility function to be strictly concave. The conditions in (3)-(4) further ensure that real money balances and consumption are normal

goods, i.e. that the utility function exhibits increasing expansion paths with respect to money and consumption.

The variable A_t describes the relevant stock of money that provides – in real terms – utility. Throughout the paper, we consider two cases, where A_t denotes money either held at the *Beginning* of the period, M_{t-1} , or at the *End* of period, M_t :¹⁴

$$A_t = \begin{cases} M_{t-1} & \text{version } B \\ M_t & \text{version } E \end{cases}.$$

The *B*-version, which, for example, relates to the money-in-the-utility function specifications in Woodford (1990), McCallum and Nelson (1999), or Lucas (2000), is consistent with Svensson's (1985) timing assumption where the goods market is closed before the asset market is opened. This means that the representative agent in period t relies on the stock of money carried over from the previous period for transactions in the goods market. In the end-of-period specification (*E*-version), which can for example be found in Brock (1974), Ljungqvist and Sargent (2000), or Woodford (2003), the stock of money held at the end of the period is assumed to provide transaction services.

Maximizing (2) subject to (1) and the no-Ponzi game condition for given initial values $M_{-1} > 0$, Z_0 and $B_{-1} \geq 0$ leads to the following first order conditions for consumption, money, labor supply, government bonds, and contingent claims:

$$\lambda_t = \begin{cases} u_c(c_t, m_{t-1}/\pi_t) & \text{version } B \\ u_c(c_t, m_t) & \text{version } E \end{cases}, \quad (5)$$

$$i_t E_t \frac{\lambda_{t+1}}{\pi_{t+1}} = \begin{cases} E_t[u_a(c_{t+1}, m_t/\pi_{t+1})/\pi_{t+1}] & \text{version } B \\ \beta^{-1} u_a(c_t, m_t) & \text{version } E \end{cases}, \quad (6)$$

$$u_l(l_t) = -w_t \lambda_t, \quad (7)$$

$$\lambda_t = \beta R_t E_t[\lambda_{t+1} \pi_{t+1}^{-1}], \quad (8)$$

$$q_{t,t+1} = \beta (\lambda_{t+1}/\lambda_t) \pi_{t+1}^{-1}, \quad (9)$$

where $i_t = R_t - 1$ denotes the net interest rate on government bonds, λ_t denotes a Lagrange multiplier, π_t the inflation rate $\pi_t = P_t/P_{t-1}$, and m_t real balances $m_t = M_t/P_t$. Equation (9) holds for each state in period $t + 1$, and determines the price of one unit of currency for a particular state at time $t + 1$ normalized by the conditional probability of occurrence of that state in units of currency in period t . Arbitrage-freeness between government bonds and contingent claims requires $R_t = 1/E_t q_{t,t+1}$. Note that beginning-of-period real balances m_{t-1} enter the set of first order conditions only in case *B*. The optimum is further characterized by the budget constraint (1) holding with equality and by the transversality condition $\lim_{i \rightarrow \infty} E_t(M_{t+i} + B_{t+i} + Z_{t+1+i}) \prod_{v=1}^i R_{t+v}^{-1} = 0$.

The final consumption good is an aggregate of differentiated goods produced by monop-

¹⁴These specifications are related, but not identical, to those applied in Carlstrom and Fuerst (2001). In particular, they assume that the monetary aggregate, which enters the utility function, is either given by $A_t = M_{t-1} + R_{t-1}B_t - B_t + P_t\tau_t$ or $A_t = M_{t-1} + R_{t-1}B_t - B_t + P_t\tau_t - P_tc_t + P_ty_t$. As a consequence, money demand can matter for local uniqueness of the equilibrium sequences of inflation and consumption under interest rate policy, even if the utility function is separable.

olistically competitive firms indexed with $i \in [0, 1]$. The CES aggregator of differentiated goods is defined as $y_t^{\frac{\epsilon-1}{\epsilon}} = \int_0^1 y_{it}^{\frac{\epsilon-1}{\epsilon}} di$, with $\epsilon > 1$, where y_t is the number of units of the final good, y_{it} the amount produced by firm i , and ϵ the constant elasticity of substitution between these differentiated goods. Let P_{it} and P_t denote the price of good i set by firm i and the price index for the final good. The demand for each differentiated good is $y_{it} = (P_{it}/P_t)^{-\epsilon} y_t$, with $P_t^{1-\epsilon} = \int_0^1 P_{it}^{1-\epsilon} di$. A firm i produces good y_i employing a technology which is linear in the labor input: $y_{it} = s_t l_{it}$, where $l_t = \int_0^1 l_{it} di$ and s_t is an i.i.d. productivity shock with mean one. Hence, labor demand satisfies: $mc_{it} = w_t/s_t$, where $mc_{it} = mc_t$ denotes real marginal costs.

We allow for a nominal rigidity in form of a staggered price setting as developed by Calvo (1983). Each period firms may reset their prices with the probability $1 - \phi$ independently of the time elapsed since the last price setting. The fraction $\phi \in [0, 1)$ of firms are assumed to adjust their previous period's prices according to the simple rule $P_{it} = \bar{\pi} P_{it-1}$, where $\bar{\pi}$ denotes the average inflation rate. Note that if $\phi > 0$, the initial price level P_{i-1} has to be known by these firms. Firms are assumed to maximize their market value, which equals the expected sum of discounted dividends $E_t \sum_{s=0}^{\infty} q_{t,t+s} D_{it+s}$, where $D_{it} \equiv (P_{it} - P_t mc_t) y_{it}$ and we used that firms also have access to contingent claims. In each period a measure $1 - \phi$ of randomly selected firms set new prices \tilde{P}_{it} as the solution to $\max_{\tilde{P}_{it}} E_t \sum_{s=0}^{\infty} \phi^s q_{t,t+s} (\bar{\pi}^s \tilde{P}_{it} y_{it+s} - P_{t+s} mc_{t+s} y_{it+s})$, s.t. $y_{it+s} = (\bar{\pi}^s \tilde{P}_{it})^{-\epsilon} P_{t+s}^{\epsilon} y_{t+s}$. The first order condition for the price of re-optimizing producers is for $\phi > 0$ given by

$$\tilde{P}_{it} = \frac{\epsilon}{\epsilon - 1} \frac{E_t \sum_{s=0}^{\infty} \phi^s [q_{t,t+s} y_{t+s} P_{t+s}^{\epsilon+1} \bar{\pi}^{-\epsilon s} mc_{t+s}]}{E_t \sum_{s=0}^{\infty} \phi^s [q_{t,t+s} y_{t+s} P_{t+s}^{\epsilon} \bar{\pi}^{(1-\epsilon)s}]} \quad (10)$$

Aggregate output is given by $y_t = (P_t^*/P_t)^{\epsilon} l_t$, where $(P_t^*)^{-\epsilon} = \int_0^1 P_{it}^{-\epsilon} di$ and thus $(P_t^*)^{-\epsilon} = \phi (\bar{\pi} P_{t-1}^*)^{-\epsilon} + (1 - \phi) \tilde{P}_t^{-\epsilon}$. If prices are flexible, $\phi = 0$, then the first order condition for the optimal price of the differentiated good reads: $mc_t = \frac{\epsilon-1}{\epsilon}$.

The public sector consists of a fiscal and a monetary authority. We consider two widely applied specifications for the monetary policy regime. The first regime is characterized by the central bank setting the nominal interest rate contingent on current or on future inflation.

$$R_t = \rho(\pi_t), \quad \text{or} \quad R_t = \rho(E_t \pi_{t+1}), \quad \text{with } \rho' \geq 0, \quad R_t \geq 1. \quad (11)$$

We disregard output (gap) as an indicator for interest rate policy, which is for example suggested by Taylor (1993). It turns out to be crucial for the impact of real balance effects on the determination of the price level and the equilibrium sequences, that the current inflation rate can be determined. Thus, for our purposes it is sufficient to focus on the cases where interest rates are set contingent either on current or on expected future inflation.

We further assume that the steady state condition $R = \pi/\beta$ has a unique solution for $R > 1$. According to the interest rate feedback rule (11), the response of the interest rate to changes in inflation, ρ_{π} , is non-negative. The second regime, is characterized by the central bank holding the money growth constant $M_t/M_{t-1} = \mu$, where $\mu \geq 1$:

$$m_t \pi_t / m_{t-1} = \mu. \quad (12)$$

The fiscal authority issues risk-free one period bonds, receives lump-sum taxes from households, and transfers from the monetary authority, such that the consolidated budget constraint reads: $R_{t-1}B_{t-1} + M_{t-1} = M_t + B_t + P_t\tau_t$. We assume that tax policy guarantees government solvency, i.e., ensures $\lim_{i \rightarrow \infty} (M_{t+i} + B_{t+i}) \prod_{v=1}^i R_{t+v}^{-1} = 0$.

3 Equilibrium determination under flexible prices

In this section, we assess how real balance effects and monetary policy affect the determination of the price level and of the rational expectations equilibrium when prices are flexible, $\phi = 0$. As described in the previous section, we consider two versions of the model which differ with regard to the stock of money that enters the utility function, i.e., the *B*-version and the *E*-version, and we consider three types of monetary policy rules described by (11) or (12). The equilibrium under flexible prices ($\phi = 0$) can then be summarized as follows.

Definition 1 *A rational expectations equilibrium (REE) is a set of sequences $\{c_t, \pi_t, a_t, R_t\}_{t=0}^{\infty}$ satisfying $u_l(c_t/s_t) = -s_t \frac{\epsilon-1}{\epsilon} u_c(c_t, a_t)$, $u_c(c_t, a_t) = \beta R_t E_t[u_c(c_{t+1}, a_{t+1}) \pi_{t+1}^{-1}]$, and $(R_t - 1)E_t[u_c(c_{t+1}, a_{t+1})/\pi_{t+1}] = E_t[u_a(c_{t+1}, a_{t+1})/\pi_{t+1}]$ for $a_t = m_{t-1}/\pi_t$ (*B*-version) or $(R_t - 1)E_t[u_c(c_{t+1}, a_{t+1})/\pi_{t+1}] = \beta^{-1} u_a(c_t, a_t)$ for $a_t = m_t$ (*E*-version), the transversality condition, and a monetary policy (11) or (12), for $\{s_t\}_{t=0}^{\infty}$, a given initial money endowment $M_{-1} > 0$, and any initial price level P_{-1} satisfying $P_{-1} > 0$.*

To identify the differences between the model versions, it is crucial to take into account that the determination of a *REE* does not require a given initial value for the price level. In the *E*-version, equilibrium sequences $\{c_t, \pi_t, a_t = A_t/P_t, R_t\}_{t=0}^{\infty}$ can be determined independently of the initial price level, such that a particular *REE* is consistent with any initial price level and thus with multiple price level sequences. In the *B*-version, the initial price level can be relevant for the equilibrium sequences due to its effect on the initial value for real balances m_{-1} , which might affect the consumption decision (see 5). Thus, the price level is non-neutral for the *REE*, such that a particular set of equilibrium sequences is associated with a unique price level sequence. This property is often summarized by the notion “nominal determinacy” (see Benhabib et al., 2001a). It is crucial to note that the role of the initial price level does not relate to the unique determination of equilibrium sequences (including the inflation sequence) which is summarized by the notion “real indeterminacy”. These properties are summarized in the following definition, which corresponds to the definition applied in Benhabib et al. (2001a).

Definition 2 *The equilibrium displays real determinacy if there exists a unique set of equilibrium sequences $\{c_t, \pi_t, a_t, R_t\}_{t=0}^{\infty}$. The equilibrium displays nominal indeterminacy if for any particular set of equilibrium sequences, there exist infinite many initial price levels P_{-1} consistent with a rational expectations equilibrium.*

A *REE*, which is characterized by real determinacy (equilibrium uniqueness) and, thus, a unique inflation sequence, can be associated with multiple price level sequences. If, for example, there are no real balance effects ($u_{ca} = 0$), the price level is neutral with regard to the determination of equilibrium sequences $\{c_t, \pi_t, R_t\}_{t=0}^{\infty}$ under interest rate policy, such that

two different values for the initial price level together with an equilibrium inflation sequence lead to two different price level sequences consistent with the *REE*. Evidently, one cannot uniquely determine a price level sequence if there are infinitely many equilibrium inflation sequences, which implies real indeterminacy.

In the following analysis, we apply Blanchard and Kahn's (1980) approach to the analysis of a rational expectations equilibrium. For this, we focus on the model's behavior in the neighborhood of the steady state, and apply a linear approximation of the set of non-linear equilibrium conditions. We, therefore, assume that the bounds on the fluctuations of the log productivity $\|\log s_t\|$ are sufficiently tight, such that s_t remains in the neighborhood of its steady state value. Throughout, we restrict our attention to *locally stable* equilibrium sequences, i.e., to equilibrium sequences that converge to the steady state.

3.1 Beginning-of-period money

We start with the case where the beginning-of-period stock of money enters the utility function. The deterministic steady state is then characterized by the following properties: $\bar{R} = \bar{\pi}/\beta$, $-u_l(\bar{c}) = u_c(\bar{c}, \bar{m}/\bar{\pi})(\epsilon - 1)/\epsilon$, and $u_c(\bar{c}, \bar{m}/\bar{\pi})(\bar{R} - 1) = u_a(\bar{m}/\bar{\pi}, \bar{c})$. A discussion of the existence and uniqueness of a steady state for $\bar{R} > 1$ can be found in appendix 6.2. Log-linearizing the model at the steady state, leads to the following set of equilibrium conditions:

$$\varepsilon_{ca}\hat{m}_{t-1} - \varepsilon_{ca}\hat{\pi}_t = (\sigma_l + \sigma_c)\hat{c}_t - (1 + \sigma_l)\hat{s}_t, \quad (13)$$

$$\sigma_c\hat{c}_t - \varepsilon_{ca}\hat{m}_{t-1} + \varepsilon_{ca}\hat{\pi}_t = \sigma_c E_t\hat{c}_{t+1} - \varepsilon_{ca}\hat{m}_t + (\varepsilon_{ca} + 1)E_t\hat{\pi}_{t+1} - \hat{R}_t, \quad (14)$$

$$(\varepsilon_{ca} + \sigma_a)\hat{m}_t = -z\hat{R}_t + (\sigma_c + \phi_{ac})E_t\hat{c}_{t+1} + (\varepsilon_{ca} + \sigma_a)E_t\hat{\pi}_{t+1}, \quad (15)$$

where $z \equiv \bar{R}/(\bar{R} - 1) > 1$, $\sigma_l \equiv \frac{\bar{l}u_{ll}}{\bar{u}_l} \geq 0$, $\sigma_c \equiv -\frac{\bar{c}u_{cc}}{\bar{u}_c} > 0$, $\sigma_a \equiv -\frac{\bar{a}u_{aa}}{\bar{u}_a} > 0$, $\varepsilon_{ca} \equiv \frac{\bar{a}u_{ca}}{\bar{u}_c} > 0$, and $\phi_{ac} \equiv \frac{\bar{c}u_{ac}}{\bar{u}_a} > 0$, and \hat{x}_t denotes the percent deviation of a generic variable x_t from its steady state value \bar{x} : $\hat{x} = \log(x_t) - \log(\bar{x})$. These conditions (and the transversality condition) have to be satisfied by the sequences for the steady state deviations of consumption, real balances, the inflation rate, and of the nominal interest rate, $\{\hat{c}_t, \hat{\pi}_t, \hat{m}_{t-1}, \hat{R}_t\}_{t=0}^\infty$ for $\{\hat{s}_t\}_{t=0}^\infty$ and a monetary policy regime satisfying

$$\hat{R}_t = \rho_\pi \hat{\pi}_t, \quad \text{or} \quad \hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}, \quad \text{or} \quad \hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t, \quad (16)$$

where ρ_π denotes the steady state inflation elasticity $\rho_\pi \equiv \rho'(\bar{\pi})(\bar{\pi}/\bar{R}) \geq 0$. It should be noted that concavity of the utility function implies: $\Upsilon \equiv \sigma_c \sigma_a - \varepsilon_{ca} \phi_{ac} > 0$,¹⁵ which restricts the magnitude of real balance effects. A closer look at the equilibrium conditions (13) and (14) reveals that the private sector behavior is not independent of the beginning-of-period value for real balances \hat{m}_{t-1} , as they are (implicitly) assumed to lower households' transactions costs. Given that \hat{m}_{t-1} is predetermined, the households' behavior can induce the economy to evolve in a history dependent way. Yet, \hat{m}_{t-1} enters the equilibrium conditions jointly with the current inflation rate. Thus, real money serves as a relevant predetermined state variable, only if the current inflation rate is determined. This further implies that the equilibrium

¹⁵We view this as a realistic implication, given that estimates of ε_{ca} and ϕ_{ac} , are usually found to be small. According to US estimates reported in Woodford (2003), ε_{ca} does not exceed 0.005 and $\phi_{ac} \leq 2$.

displays nominal determinacy.

Proposition 1 *The equilibrium displays nominal determinacy, if the beginning-of-period stock of money enters the utility function.*

Monetary policy is further decisive for real determinacy and, thus, for the possibility to uniquely determine a price level sequence. In the subsequent analysis, we will show that this requires the central bank to set the nominal interest rate contingent on current inflation. Under an interest rate peg, $\rho_\pi = 0$, an inflation sequence and, therefore, a price level sequence cannot be uniquely determined.¹⁶

We, firstly, examine the case where the central bank sets the nominal interest rate according to an interest rate feedback rule. At first, we consider current inflation as the policy indicator, $\hat{R}_t = \rho_\pi \hat{\pi}_t$. The following proposition summarizes the equilibrium properties for the cases where the labor supply elasticity $1/\sigma_l$ takes a finite value or is infinite.

Proposition 2 (*B, Interest rate policy, $\hat{R}_t = \rho_\pi \hat{\pi}_t$*) *Consider that beginning-of-period money enters the utility function and that the nominal interest rate is set contingent on changes in current inflation $\hat{R}_t = \rho_\pi \hat{\pi}_t$.*

1. *When the labor supply elasticity is finite, $\sigma_l > 0$, the equilibrium displays real determinacy and local stability if and only if*

- (a) *$\overline{\rho_{\pi 1}} < \rho_\pi < 1$ for $\varepsilon_{ca} > \frac{\sigma_a}{2z-1}$ and $\sigma_l > \overline{\sigma_l}$, leading to non-oscillatory equilibrium sequences, or $\rho_\pi \in (1, \overline{\rho_{\pi 1}})$, leading to oscillatory equilibrium sequences,*
- (b) *$\rho_\pi > 1$ for $\varepsilon_{ca} < \frac{\sigma_a}{2z-1}$ or $\sigma_l < \overline{\sigma_l}$, leading to oscillatory equilibrium sequences,*

where $\overline{\rho_{\pi 1}} \equiv \frac{\sigma_l(\varepsilon_{ca} + \sigma_a) + \Upsilon}{\sigma_l(2z-1)\varepsilon_{ca} - \sigma_l\sigma_a - \Upsilon}$ and $\overline{\sigma_l} \equiv \frac{\Upsilon}{(2z-1)\varepsilon_{ca} - \sigma_a}$.

2. *When the labor supply elasticity is infinite, $\sigma_l = 0$, consumption \hat{c}_t cannot uniquely be determined, while the equilibrium sequences $\{E_t \hat{c}_{t+1}, \hat{\pi}_t, \hat{m}_t, \hat{R}_t\}_{t=0}^\infty$ are locally stable and uniquely determined if and only if $\rho_\pi > 1$.*

Proof. See appendix 6.3.

Proposition 2 reveals that the requirements for local equilibrium stability and uniqueness in terms of the policy parameter ρ_π are not robust with regard to changes in the elasticities ε_{ca} and σ_l .¹⁷ For finite labor supply elasticities, $\sigma_l > 0$, passiveness is necessary for locally stable, unique, and non-oscillatory equilibrium sequences (see part 1a). An interest rate peg, however, violates the conditions in part 1 of proposition 2 and, thus, implies real indeterminacy. Under an active interest rate policy, locally stable and unique equilibrium sequences are oscillatory, which is hardly recommendable for a central bank that aims at stabilizing

¹⁶It should further be noted that a *REE* displays nominal indeterminacy if there are no real balance effects, $\varepsilon_{ca} = \phi_{ac} = 0$. Nevertheless, one can always compute a price level sequence for a particular initial price level and a sequence of inflation rates.

¹⁷Note that for the sets $(\overline{\rho_{\pi 1}}, 1)$ and $(1, \overline{\rho_{\pi 1}})$ (see part 1a. of proposition 2) to be non-empty $\sigma_l > \Upsilon[(z-1)\varepsilon_{ca} - \sigma_a]^{-1}$ and $\varepsilon_{ca} > \sigma_a/(z-1)$, and, respectively, $\sigma_l < \Upsilon[(z-1)\varepsilon_{ca} - \sigma_a]^{-1}$ or $\varepsilon_{ca} < \sigma_a/(z-1)$ has to be satisfied.

the economy. Thus, when beginning-of-period money relates to households' consumption, interest rate policy that reacts on current inflation should rather be passive than active for macroeconomic stability and for the unique determination of the price level.

To see this, suppose that inflation exceeds its steady state value and equilibrium sequences are non-oscillatory.¹⁸ Given that the inflation elasticity is positive, $\rho_\pi > 0$, the nominal interest rate rises, which – ceteris paribus – causes households to reduce their end-of-period real money holdings \hat{m}_t , by (15). According to (14), the expected real interest rate is further negatively related to the growth rate of real balances. Thus, an active interest rate setting, $\rho_\pi > 1$, leads to a decline in the level and the growth rate of real balances, such that the sequences of real balances and, thus, of consumption and inflation do not converge to the steady state.

Notably, the equilibrium exhibits different properties for an infinite labor supply elasticity, $\sigma_l = 0$ (see part 2 of proposition 2). In this case, the amount of labor supplied by the households is not related to their consumption expenditures, and the marginal utility of consumption is exogenously determined (see 13): $\hat{h}_t = -\hat{s}_t$, where $\hat{h}_t = \varepsilon_{ca}\hat{m}_{t-1} - \sigma_c\hat{c}_t - \varepsilon_{ca}\hat{\pi}_t$. Hence, (14) and (15) reduce to $\hat{s}_t - E_t\hat{s}_{t+1} = E_t\hat{\pi}_{t+1} - \hat{R}_t$ and $\sigma_a\hat{m}_t = -z\hat{R}_t + \phi_{ac}E_t\hat{c}_{t+1} + \sigma_a E_t\hat{\pi}_{t+1} + E_t\hat{s}_{t+1}$, implying that the equilibrium is not associated with a unique value for beginning-of-period real money and that current consumption cannot be determined. The equilibrium sequences for $E_t\hat{c}_{t+1}$, $\hat{\pi}_t$, \hat{m}_t , and \hat{R}_t are then locally stable and uniquely determined for an active interest rate policy, which contrasts the results for the case of finite labor supply elasticities, $\sigma_l > 0$, presented in part 1 of proposition 2.

We now turn to the case where the central bank applies a forward looking rule, $\hat{R}_t = \rho_\pi E_t\hat{\pi}_{t+1}$.

Proposition 3 (B, Interest rate policy, $\hat{R}_t = \rho_\pi E_t\hat{\pi}_{t+1}$) *Consider that beginning-of-period money enters the utility function and that the nominal interest rate is set contingent on changes in future inflation $\hat{R}_t = \rho_\pi E_t\hat{\pi}_{t+1}$. Then, consumption and inflation cannot uniquely be determined.*

1. *When the labor supply elasticity is finite, $\sigma_l > 0$, then $\rho_\pi > 1$ is a necessary condition for uniqueness and local stability of the equilibrium sequences $\{E_t\hat{c}_{t+1}, E_t\hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\}_{t=0}^\infty$. Necessary and sufficient conditions are given by:*

- (a) $1 < \rho_\pi$ for $\sigma_l > \overline{\sigma_{l2}}$ and $\varepsilon_{ca} > \frac{\sigma_a}{z-1}$,
- (b) $1 < \rho_\pi < \overline{\rho_{\pi 2}}$, for $\sigma_l < \overline{\sigma_{l2}}$ or $\varepsilon_{ca} < \frac{\sigma_a}{z-1}$, or $1 < \overline{\rho_{\pi 2}} < \rho_\pi$ if $\sigma_l > \overline{\sigma_l}$ and $\varepsilon_{ca} > \frac{\sigma_a}{2z-1}$, for $\sigma_l \in (\overline{\sigma_l}, \overline{\sigma_{l2}})$ or $\varepsilon_{ca} \in (\frac{\sigma_a}{2z-1}, \frac{\sigma_a}{z-1})$,
- (c) $1 < \overline{\rho_{\pi 2}} < \rho_\pi < -\overline{\rho_{\pi 1}}$ for $\sigma_l < \overline{\sigma_l}$ or $\varepsilon_{ca} < \frac{\sigma_a}{2z-1}$,

where $\overline{\sigma_{l2}} \equiv \frac{\Upsilon}{(z-1)\varepsilon_{ca}-\sigma_a}$ and $\overline{\rho_{\pi 2}} \equiv \frac{\Upsilon+\sigma_l(\varepsilon_{ca}+\sigma_a)}{\Upsilon+\sigma_l(\varepsilon_{ca}+\sigma_a)-z\varepsilon_{ca}\sigma_l}$.

2. *When the labor supply elasticity is infinite, $\sigma_l = 0$, then the equilibrium sequences $\{E_t\hat{c}_{t+1}, E_t\hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\}_{t=0}^\infty$ are locally stable and uniquely determined if and only if $\rho_\pi \neq 1$.*

¹⁸The latter property implies that current and expected future inflation are not negatively related.

Proof. See appendix 6.4.

Compared to proposition 2 the most fundamental difference relates to the role of the beginning-of-period real balances, \hat{m}_{t-1} . It turns out that the households' consumption decision is not affected by the latter if inflation is indeterminated, such as under an interest rate peg or a forward looking interest rate rule. In these cases, there are infinitely many values for the current inflation rate that are consistent with the equilibrium conditions. Given that \hat{m}_{t-1} enters the latter jointly with the current inflation rate, there are also multiple admissible values for beginning-of-period real balances \hat{m}_{t-1} . If, however, current inflation serves as the policy indicator, it can be determined in every period, such that \hat{m}_{t-1} serves as a relevant determinant for current consumption by the equilibrium condition (13). As a consequence, the initial price level P_{-1} (and, thus, the initial stock of money $m_{-1} = M_{-1}/P_{-1}$), matters for the equilibrium allocation. If, however, the central bank sets the nominal interest rate contingent on expected future inflation, one can only determine the expected future values but not the current values for inflation and consumption. Subsequent to the initial period, households will adjust \hat{m}_t in accordance with their desired future consumption $E_t \hat{c}_{t+1}$, so that their behavior is not history dependent. Thus, the beginning-of period real value of money is then irrelevant, and there are multiple price level sequences (though the equilibrium displays nominal determinacy).

For equilibrium sequences to be locally stable and unique the inflation elasticity ρ_π has to satisfy certain conditions.¹⁹ Under an interest rate rule featuring current inflation, it turns out that there is no robust value for the inflation elasticity that ensures local stability and uniqueness. For example, when the real balance effect and the labor supply elasticity satisfy $\varepsilon_{ca} > \frac{\sigma_a}{2z-1}$ and $\sigma_l > \frac{\Upsilon}{(2z-1)\varepsilon_{ca}-\sigma_a}$, interest rate policy should be passive, $\rho_\pi < 1$, while the inverse, $\rho_\pi > 1$, is required under $\varepsilon_{ca} < \frac{\sigma_a}{2z-1}$ or $\sigma_l < \bar{\sigma}_l$ (see proposition 2). When the central bank sets the nominal interest rate contingent on expected future inflation, activeness $\rho_\pi > 1$ is always necessary (but not sufficient) for uniqueness. As in the previous case (see part 2 of proposition 2), the equilibrium exhibits different properties if the labor supply elasticity is infinite $\sigma_l = 0$ as described in part 2 of proposition 3. With a forward looking interest rate rule, the model then reduces to a set of static equilibrium conditions characterized by unique equilibrium sequences $\{E_t \hat{c}_{t+1}, E_t \hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\}_{t=0}^\infty$ for any non-zero inflation elasticity $\rho_\pi \neq 1$.

Under a money growth regime equilibrium determination is less sensitive. Ruling out unreasonable parameter values, we focus, for convenience, on the case where the inverse of the elasticity of intertemporal substitution of money is not extremely large, $\sigma_a < z = \bar{R}/(\bar{R}-1)$.²⁰

Proposition 4 (B, Money growth policy) *Suppose that beginning-of-period money enters the utility function and that $\sigma_a < z$. Under a constant money growth rule, the equilibrium sequences $\{\hat{c}_t, \hat{\pi}_t, \hat{m}_{t-1}\} \forall t \geq 1$ and $\{\hat{R}_t\} \forall t \geq 0$, are locally stable and uniquely determined, and there exists a unique consistent price level $\forall t \geq 0$.*

Proof. See appendix 6.5.

¹⁹Non-emptiness of the sets for ρ_π requires $\bar{\rho}_{\pi 2} > 1$ and $-\bar{\rho}_{\pi 1} > \bar{\rho}_{\pi 2}$, which is fulfilled for the given restrictions on σ_l and ε_{ca} in part 1b and 1c.

²⁰It should be noted that $\sigma_a < z$ is just a sufficient precondition for the result in proposition 4 and hardly restrictive if one assigns values for σ_a that relate to reasonable magnitudes of σ .

A comparison of the results in the propositions 2-4 shows that the money growth regime leads to an equilibrium behavior being different from the behavior under both interest rate policy regimes. On the one hand, the price level can always be determined if real balances are determined, given that the value for the nominal stock of money is known in every period. On the other hand, the initial values for the inflation rate $\hat{\pi}_0$ and real money \hat{m}_{-1} are irrelevant for equilibrium determination, implying that there are – for different initial price levels – multiple values for both which are consistent with a unique set of equilibrium sequences $\{\hat{c}_t, \hat{\pi}_t, \hat{m}_{t-1}\}_{t=1}^{\infty}$ and $\{\hat{R}_t\}_{t=0}^{\infty}$. As a consequence, the *REE* does not feature real balances as a relevant endogenous state variable. Put differently, for the economy to evolve in a history dependent way, it is, therefore, not sufficient that monetary policy is conducted in a backward looking way. In fact, it is the households' consumption decision rather than a restriction on the evolution of money, which is responsible for the equilibrium sequences to depend on beginning-of-period money holdings. There is an analogy to the role of physical capital in a standard real business cycle model with a depreciation rate equal to one. Capital remains a relevant state variable, even though the model (virtually) lacks an accumulation equation.²¹

3.2 End-of-period money

Next, we will briefly summarize the requirements for equilibrium determination under the assumption that end-of-period money holdings enter the utility function. This case has also been examined by Benhabib et al. (2001a) and by Woodford (2003) for interest rate policies, and by Carlstrom and Fuerst (2003) for money growth rules. The deterministic steady state for this version is characterized by the following conditions, $\bar{R} = \bar{\pi}/\beta$, $-u_l(\bar{c}) = u_c(\bar{c}, \bar{m})(\epsilon - 1)/\epsilon$, and $u_c(\bar{c}, \bar{m})(R - 1) = u_a(\bar{m}, \bar{c})$.²² Log-linearizing the model summarized in definition 1 for $A_t = M_t$ at the steady state with $\bar{R} > 1$ leads to the following set of equilibrium conditions:

$$\varepsilon_{ca}\hat{m}_t = (\sigma_l + \sigma_c)\hat{c}_t - (1 + \sigma_l)\hat{s}_t, \quad (17)$$

$$\sigma_c\hat{c}_t - \varepsilon_{ca}\hat{m}_t = \sigma_c E_t\hat{c}_{t+1} - \varepsilon_{ca}E_t\hat{m}_{t+1} - \hat{R}_t + E_t\hat{\pi}_{t+1}, \quad (18)$$

$$(\varepsilon_{ca} + \sigma_a)\hat{m}_t = (\phi_{ac} + \sigma_c)\hat{c}_t - (z - 1)\hat{R}_t. \quad (19)$$

The conditions (17)-(19), the transversality condition, and a monetary policy rule (16) have to be satisfied by sequences $\{\hat{c}_t, \hat{\pi}_t, \hat{m}_t, \hat{R}_t\}_{t=0}^{\infty}$ for $\{\hat{s}_t\}_{t=0}^{\infty}$. In contrast to the *B*-version, consumption and inflation are independent of beginning-of-period real balances. Put differently, the private sector behavior is entirely forward-looking in the *E*-version with the consequence that the equilibrium displays nominal indeterminacy.

Proposition 5 *The equilibrium displays nominal indeterminacy, if the end-of-period stock of money enters the utility function.*

²¹Consider a real version of our model, with perfect competition, a production technology satisfying $y_t = s_t k_{t-1}^\alpha l_t^{1-\alpha}$, where k_{t-1} denotes the beginning-of-period stock of physical capital and $\alpha \in (0, 1)$, and a capital depreciation rate of 100%. Nevertheless, capital serves as a relevant state variable, i.e., k_{t-1} affects the equilibrium allocation in period t .

²²A discussion of steady state uniqueness is provided in appendix 6.2.

The following proposition summarizes the conditions for equilibrium determination under interest rate policy.

Proposition 6 (*E*, Interest rate policy) *Suppose that end-of-period money enters the utility function and that the central bank sets the nominal interest rate.*

1. *When current inflation enters the interest rate rule, $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$, the equilibrium displays real determinacy and local stability if and only if $\rho_\pi > 1$.*
2. *When future inflation enters the interest rate rule, $\widehat{R}_t = \rho_\pi E_t \widehat{\pi}_{t+1}$, inflation cannot uniquely be determined. The equilibrium sequences $\{\widehat{c}_t, E_t \widehat{\pi}_{t+1}, \widehat{m}_t, \widehat{R}_t\}_{t=0}^\infty$ are locally stable and uniquely determined if and only if i) $\rho_\pi > 1$ or $\rho_\pi < \left(1 + \frac{2(z-1)\sigma_l \varepsilon_{ca}}{\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)}\right)^{-1}$ for $\sigma_l > 0$, and ii) $\rho_\pi \neq 1$ for $\sigma_l = 0$.*

Proof. See appendix 6.6.

As in the *B*-version, equilibrium determination depends on the particular interest rate rule. When the nominal interest rate is set contingent on current inflation, inflation can be determined for all periods. Under a forward looking interest rate policy, one can only uniquely determine expected future inflation. In any case, the initial price level and initial real balances are irrelevant for a *REE*, implying nominal indeterminacy and the absence of an endogenous state variable. Uniqueness of equilibrium sequences is further ensured by an active interest rate policy, $\rho_\pi > 1$, under both types of rules. For the special case, where the labor supply elasticity is infinite, any forward looking interest rate rule satisfying $\rho_\pi \neq 1$ leads to unique equilibrium sequences $\{\widehat{c}_t, E_t \widehat{\pi}_{t+1}, \widehat{m}_t, \widehat{R}_t\}_{t=0}^\infty$. Turning to the case where the central bank holds the money growth rate constant, we find that the equilibrium behavior closely relates to the one in the *B*-version.

Proposition 7 (*E*, Money growth policy) *Suppose that end-of-period money enters the utility function and that the money growth rate is held constant. Then, the equilibrium sequences $\{\widehat{c}_t, \widehat{m}_t, \widehat{R}_t\} \forall t \geq 0$ and $\{\widehat{\pi}_t\} \forall t \geq 1$ are locally stable and uniquely determined, and there exists a unique consistent price level $\forall t \geq 0$.*

Proof. See appendix 6.7.

To summarize, the specification of money demand has substantial consequences for the determination of equilibrium sequences and for macroeconomic stability. The beginning-of-period value for real money balances is only relevant for equilibrium determination in the *B*-version under a non-forward looking interest rate rule. In the *E*-version, where the households' behavior lacks any backward looking element, the initial value of real balances is irrelevant for any policy regime under consideration. Whether beginning-of period real money is serving as a relevant endogenous state variable or not, is, on the one hand, decisive for a unique determination of a price level, and, on the other hand, crucially affects the conditions for local stability and uniqueness under an interest rate policy regime: Policy should rather be passive than active, to avoid unstable or oscillatory equilibrium sequences. Under a constant money

growth regime, however, real determinacy and uniqueness of a price level sequence is ensured for both versions. In particular, consumption is determined regardless of the labor supply elasticity. This contrasts the findings under interest rate policy, where current consumption and, thus, household welfare cannot uniquely be determined if the labor supply elasticity is infinite, or if interest rates are set contingent on changes in expected future inflation.

3.3 Related results

The main novel results in this section refer to the case where beginning-of-period money enters the utility function and the central bank applies an interest rate rule, while some results for the alternative cases correspond to results in related studies on real balances effects and equilibrium determinacy in flexible price models. For example, our findings for the *E*-version (see part 1 of proposition 6) resemble the results in Benhabib et al. (2001a) and Woodford (2003) for non-separable utility functions. They find that when current inflation serves as an indicator, active interest rate setting is necessary and sufficient for local stability and uniqueness. This, however, changes when beginning-of-period money provides utility, since equilibrium sequences are then – except for the case $\sigma_l = 0$ – unstable or oscillatory (see proposition 2). Thus, the literature has disregarded the role of predetermined real balances as a relevant state variable, which substantially affects the real and nominal determinacy properties.

If the monetary authority applies a constant money growth rule, then local stability and uniqueness impose restrictions on preferences only in case where the stock of money held at the beginning of the period provides utility. In particular, the inverse of the intertemporal elasticity of substitution for real money balances should not be too large (see proposition 4), which corresponds to the results in Brock (1974), Matsuyama (1990), Carlstrom and Fuerst (2003), and Woodford (2003). Assuming that end-of-period money provides transaction services, Brock (1974), Matsuyama (1990), and Woodford (2003), show that local stability and uniqueness is ensured if consumption and real balances are Edgeworth-complements, as in our framework. Furthermore, Carlstrom and Fuerst (2003) find that the intertemporal elasticity of substitution for money can matter for local stability and uniqueness is guaranteed, as in proposition 4.

To unveil the role of non-separability for the results and to facilitate comparisons with related studies (see, e.g., Carlstrom and Fuerst, 2001), we further briefly discuss the case where money demand is separable, $\varepsilon_{ca} = \phi_{ac} = 0$. Then, the model reduces to

$$\hat{R}_t = E_t \hat{\pi}_{t+1}, \text{ and } \sigma_a \hat{m}_t = \begin{cases} -(z - \sigma_a) \hat{R}_t & \text{for the } B\text{-version} \\ -(z - 1) \hat{R}_t & \text{for the } E\text{-version} \end{cases}.$$

while consumption is exogenously determined. When utility is separable, the conditions for uniqueness under money growth policy, which are presented in proposition 4 and 7, are unchanged. In contrast to the results for the non-separable case, the particular stock of money that enters the utility function is now irrelevant for equilibrium determination under interest rate policy: Equilibrium uniqueness requires $\rho_\pi > 1$ for $\hat{R}_t = \rho_\pi \hat{\pi}_t$ and $\rho_\pi \neq 1$ for $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$, which accords to the results in Carlstrom and Fuerst (2001). As in the case of non-separable utility, current inflation cannot be determined under a forward looking interest

rate rule, while under a money growth rule inflation is only indetermined in the first period.

4 Staggered Price Setting

In this section, the conditions for local stability and uniqueness of a rational expectations equilibrium under staggered price setting are derived for the same versions as in the previous section. The equilibrium under sticky prices ($\phi > 0$) can be summarized as follows.

Definition 3 *A rational expectations equilibrium is a set of sequences $\{c_t, l_t, \lambda_t, \pi_t, P_t^*, P_t, \tilde{P}_t, mc_t, w_t, a_t, R_t\}_{t=0}^\infty$, where $a_t = m_{t-1}/\pi_t$ (B-version) or $a_t = m_t$ (E-version), satisfying the firms' first order conditions $w_t = s_t mc_t$, (10) where $\tilde{P}_{it} = \tilde{P}_t$, and $P_t^{1-\epsilon} = \phi(\bar{\pi}P_{t-1})^{1-\epsilon} + (1-\phi)\tilde{P}_t^{1-\epsilon}$, the households' first order conditions (5)-(8) and $\pi_t = P_t/P_{t-1}$, the aggregate resource constraint $c_t = (P_t^*/P_t)^\epsilon s_t l_t$, and $(P_t^*)^{-\epsilon} = \phi(\bar{\pi}P_{t-1}^*)^{-\epsilon} + (1-\phi)\tilde{P}_t^{-\epsilon}$, the transversality condition, and monetary policy (11) or (12), for $\{s_t\}_{t=0}^\infty$ and given initial values $P_{-1} > 0$, $P_{-1}^* > 0$, and $m_{-1} = M_{-1}/P_{-1} > 0$.*

Note that in the version with staggered price setting the initial price indices P_{-1} and P_{-1}^* add to the set of given initial values, which under flexible prices only consists of initial cash balances M_{-1} . Thus, there is a unique sequence of aggregate prices $\{P_t\}_{t=0}^\infty$, whenever the current inflation rate π_t is uniquely determined. We will, therefore, not refer to the issue of price level determination in what follows.

4.1 Beginning-of-period money

As in the previous section, we firstly consider the B-version where beginning-of-period money provides utility, $A_t = M_{t-1}$. Log-linearizing the equilibrium conditions summarized in definition 3 at the steady state with $\bar{R} > 1$ (see appendix 6.2) leads to the following set of conditions

$$\hat{\pi}_t - \beta E_t \hat{\pi}_{t+1} = \chi[(\sigma_l + \sigma_c)\hat{c}_t - \varepsilon_{ca}\hat{m}_{t-1} + \varepsilon_{ca}\hat{\pi}_t] - \chi(1 + \sigma_l)\hat{s}_t, \quad (20)$$

$$\sigma_c \hat{c}_t - \varepsilon_{ca}\hat{m}_{t-1} + \varepsilon_{ca}\hat{\pi}_t = \sigma_c E_t \hat{c}_{t+1} - \varepsilon_{ca}\hat{m}_t + (\varepsilon_{ca} + 1)E_t \hat{\pi}_{t+1} - \hat{R}_t, \quad (21)$$

$$(\varepsilon_{ca} + \sigma_a)\hat{m}_t = -z\hat{R}_t + (\sigma_c + \phi_{ac})E_t \hat{c}_{t+1} + (\varepsilon_{ca} + \sigma_a)E_t \hat{\pi}_{t+1}, \quad (22)$$

where $\chi = (1 - \phi)(1 - \beta\phi)\phi^{-1} > 0$ (see Yun, 1996). The conditions (20)-(22) have together with the transversality condition and a monetary policy rule (16) to be satisfied by sequences $\{\hat{c}_t, \hat{\pi}_t, \hat{m}_{t-1}, \hat{R}_t\}_{t=0}^\infty$ for $\{\hat{s}_t\}_{t=0}^\infty$ and a given initial value \hat{m}_{-1} . It should be noted that inflation tends, by the aggregate supply relation (20), to decrease with real money balances. Given that consumption and money are Edgeworth-complements, marginal utility of consumption increases with real balances. As households seek to equalize the marginal utility of consumption and leisure (see (5) and (7)), demand for leisure decreases, which leads to a shift of the labor supply curve towards lower real wages. Hence, a rise in real balances tends to lower real marginal costs, inducing firms to cut their prices.

The following proposition summarizes the conditions for local stability and uniqueness for the case where the central bank sets the nominal interest rate contingent on changes in the current inflation rate, $\hat{R}_t = \rho_\pi \hat{\pi}_t$.

Proposition 8 (*B, Interest rate policy, $\hat{R}_t = \rho_\pi \hat{\pi}_t$, $\phi > 0$*) Suppose that prices are not perfectly flexible, $\phi > 0$, that beginning-of-period money enters the utility function, and that the central bank sets the nominal interest rate contingent on changes in current inflation, $\hat{R}_t = \rho_\pi \hat{\pi}_t$.

1. When the labor supply elasticity is finite, $\sigma_l > 0$, the equilibrium displays real determinacy and local stability,
 - (a) if and only if $\bar{\rho}_{\pi 3} < \rho_\pi < 1$ for $\varepsilon_{ca} > \frac{\sigma_a}{(2z-1)}$ and $\sigma_l > \bar{\sigma}_l$, leading to non-oscillatory equilibrium sequences, or $\max \{1, \beta \Upsilon(z\sigma_l \chi \varepsilon_{ca})^{-1}\} < \rho_\pi < \bar{\rho}_{\pi 3}$, leading to oscillatory equilibrium sequences, or
 - (b) if $\max \{1, \beta \Upsilon(z\sigma_l \chi \varepsilon_{ca})^{-1}\} < \rho_\pi$ for $\varepsilon_{ca} > \frac{\sigma_a}{(2z-1)}$ and $\sigma_l < \bar{\sigma}_l$, or $\varepsilon_{ca} < \frac{\sigma_a}{(2z-1)}$, leading to oscillatory equilibrium sequences,

where $\bar{\rho}_{\pi 3} \equiv \frac{\sigma_l \chi (\varepsilon_{ca} + \sigma_a) + \Upsilon(2\beta + \chi + 2)}{\chi[\sigma_l(2z-1)\varepsilon_{ca} - \sigma_l \sigma_a - \Upsilon]}$ and $\bar{\sigma}_l \equiv \frac{\Upsilon}{(2z-1)\varepsilon_{ca} - \sigma_a}$.

2. When the labor supply elasticity is infinite, $\sigma_l = 0$, the equilibrium displays real determinacy and local stability if and only if $\rho_\pi > 1$.

Proof. See appendix 6.8.

As summarized in part 1 of proposition 8, passive interest rate setting $\rho_\pi < 1$ is necessary for locally stable, unique, and non-oscillatory equilibrium sequences, if the labor supply elasticity is finite $\sigma_l > 0$.²³ As in the corresponding case with flexible prices (see part 1 of proposition 2), real money is a relevant state variable, such that the economy evolves in a history dependent way. Hence, the central bank should refrain from setting the nominal interest rate in an active way. Under separable preferences, an active interest rate policy, which leads to an unstable eigenvalue, is required for local equilibrium uniqueness in the absence of a relevant endogenous predetermined state variable (see Woodford, 2003). Here, the existence of the latter is responsible for an active interest rate policy to destabilize the economy. As shown by Dupor (2001), an analogous result can arise when physical capital is introduced in a sticky price model. Notably, beginning-of-period real balances are – as in the corresponding case under flexible prices – irrelevant, when the labor supply elasticity is infinite, $\sigma_l = 0$.

The following proposition summarizes the results for a forward looking interest rate rule, $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$.

Proposition 9 (*B, Interest rate policy, $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$, $\phi > 0$*) Suppose that prices are not perfectly flexible, $\phi > 0$, that beginning-of-period money enters the utility function, and that the central bank sets the nominal interest rate contingent on changes in expected future inflation, $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$. The equilibrium displays real determinacy and local stability if and only if $\rho_\pi < 1$, leading to non-oscillatory equilibrium sequences, or if $\rho_\pi > -\bar{\rho}_{\pi 3}$ and $\sigma_l < \bar{\sigma}_l$ or $\varepsilon_{ca} < \frac{\sigma_a}{2z-1}$, leading to oscillatory equilibrium sequences, where $-\bar{\rho}_{\pi 3} \equiv \frac{\Upsilon(2+2\beta+\chi)+\chi\sigma_l(\sigma_a+\varepsilon_{ca})}{\chi(\Upsilon+\sigma_l((1-2z)\varepsilon_{ca}+\sigma_a))} > 1$. Otherwise, the equilibrium sequences are unstable.

²³Non-emptiness of the sets $(\bar{\rho}_{\pi 3}, 1)$ and $(\max \{1, \beta \Upsilon(z\sigma_l \chi \varepsilon_{ca})^{-1}\}, \bar{\rho}_{\pi 3})$, requires $\sigma_l > \frac{\Upsilon(\beta+\chi+1)}{(z-1)\varepsilon_{ca}-\sigma_a}$ and $\varepsilon_{ca} > \frac{\sigma_a}{z-1}$, and, respectively, $\sigma_l < \frac{\Upsilon(\beta+\chi+1)}{(z-1)\varepsilon_{ca}-\sigma_a}$ or $\varepsilon_{ca} < \frac{\sigma_a}{z-1}$.

Proof. See appendix 6.9.

According to the conditions presented in proposition 9, passiveness is necessary and sufficient for the existence of exactly one stable and positive eigenvalue, if future inflation serves as the indicator for interest rate policy. In contrast to the former case, this result does not depend on the labor supply elasticity σ_l (see proposition 8). If interest rate policy is extremely reactive, $\rho_\pi > \max\{1, -\overline{\rho_{\pi 3}}\}$, equilibrium sequences can also be locally stable and unique, though they are oscillatory in this case. For any other inflation elasticity, the central bank induces the equilibrium sequences to be unstable.²⁴

Now, suppose that the central bank conducts monetary policy according to a constant money growth rule. The following proposition summarizes the local equilibrium properties, where we focus, for simplicity, on the case where the intertemporal substitution elasticity of money does not take unreasonably large values, $\sigma_a < 2z$ (see proposition 4 for a related assumption).

Proposition 10 (*B, Money growth policy, $\phi > 0$*) *Suppose that prices are not perfectly flexible, $\phi > 0$, and that beginning-of-period money enters the utility function, which satisfies $\sigma_a < 2z$. Then, under a constant money growth policy the equilibrium displays real determinacy and local stability if but not only if $\phi_{ac} < (z - 1)\sigma_c$. The equilibrium sequences are then non-oscillatory.*

Proof. See appendix 6.10.

As described in proposition 10, a constant money growth policy can ensure local stability and uniqueness if real balance effects are not too large, $\phi_{ac} < (z - 1)\sigma_c$. In contrast to the corresponding case with flexible prices (see proposition 4), current inflation can – due to sticky prices – be determined, implying that beginning-of-period real balances now serve as a relevant state variable. Thus, local stability demands a stable eigenvalue, which requires the equilibrium sequence of real balances to be positively related to inflation, since constant money growth implies a negative inflation feedback. Even in the presence of the above mentioned negative partial impact of real balances on current inflation via the aggregate supply relation (20), such a positive relation arises if the real balance effect is not extremely large. In this case, for which $\phi_{ac} < (z - 1)\sigma_c$ is sufficient, local stability and uniqueness arises. This, will be discussed in more detail in the last part of this section.

4.2 End-of-period money

Next, we examine the *E*-version where the end-of-period stock of money enters the utility function. The main difference to the former case can immediately be seen from the following

²⁴Note that if $\sigma_l = 0$, the threshold $-\overline{\rho_{\pi 3}}$ reduces to the term $1 + \frac{2(1+\beta)}{\chi}$, which also serves as an upper bound for the inflation elasticity in the corresponding model with a separable utility function (see Carlstrom and Fuerst, 2001).

set of linearized equilibrium conditions (see definition 3):

$$\widehat{\pi}_t - \beta E_t \widehat{\pi}_{t+1} = \chi [(\sigma_l + \sigma_c) \widehat{c}_t - \varepsilon_{ca} \widehat{m}_t] - \chi (1 + \sigma_l) \widehat{s}_t, \quad (23)$$

$$\sigma_c \widehat{c}_t - \varepsilon_{ca} \widehat{m}_t = \sigma_c E_t \widehat{c}_{t+1} - \varepsilon_{ca} E_t \widehat{m}_{t+1} - \widehat{R}_t + E_t \widehat{\pi}_{t+1}, \quad (24)$$

$$(\varepsilon_{ca} + \sigma_a) \widehat{m}_t = (\phi_{ac} + \sigma_c) \widehat{c}_t - (z - 1) \widehat{R}_t. \quad (25)$$

The conditions (23)-(25) together with the transversality condition and a monetary policy rule (16) have to be satisfied by sequences $\{\widehat{c}_t, \widehat{\pi}_t, \widehat{m}_t, \widehat{R}_t\}_{t=0}^{\infty}$ for $\{\widehat{s}_t\}_{t=0}^{\infty}$ and a given initial value \widehat{m}_{-1} . As beginning-of-period real balances do not enter the set of equilibrium conditions (23)-(25), the solution does, evidently, not depend on the realizations of a predetermined variable unless policy is history dependent. Put differently, the households' behavior is entirely forward looking such that history dependence can only be due to monetary policy. The following proposition summarizes the local equilibrium properties for both interest rate rules.

Proposition 11 (*E*, Interest rate policy, $\phi > 0$) *Suppose that prices are not perfectly flexible, $\phi > 0$, that end-of-period money enters the utility function, and that the central bank sets the nominal interest rate.*

1. *When current inflation serves as the indicator, $\widehat{R}_t = \rho_{\pi} \widehat{\pi}_t$, the equilibrium displays real determinacy and local stability if and only if $\rho_{\pi} > 1$.*
2. *When future inflation serves as the indicator, $\widehat{R}_t = \rho_{\pi} E_t \widehat{\pi}_{t+1}$, the equilibrium displays real determinacy and local stability if and only if $1 < \rho_{\pi} < \min\{\overline{\rho_{\pi 4}}, \overline{\rho_{\pi 5}}\}$ or $\max\{1, \overline{\rho_{\pi 4}}\} < \rho_{\pi} < \overline{\rho_{\pi 5}}$, where $\overline{\rho_{\pi 4}} \equiv \frac{\beta \Upsilon}{\chi \sigma_l \varepsilon_{ca} (z-1)} \geq 0$ and $\overline{\rho_{\pi 5}} \equiv \frac{\Upsilon(2+2\beta+\chi) + \chi \sigma_l (\sigma_a + \varepsilon_{ca})}{\chi \sigma_l ((2z-1)\varepsilon_{ca} + \sigma_a) + \chi \Upsilon} \geq 0$, for $\sigma_l > 0$, and $\rho_{\pi} \in (1, 1 + 2(1 + \beta)/\chi)$ for $\sigma_l = 0$.*

Proof. See appendix 6.11.

According to proposition 11, active interest rate setting is necessary (but not sufficient) for real determinacy.²⁵ When the central bank applies a forward looking rule, it should not raise the interest rate too aggressively with inflation (see part 2 of proposition 11).²⁶ Thus, non-separability does for the *E*-version not lead to a fundamental deviation from the principles for equilibrium determinacy in the corresponding model with a separable utility function. Compared to the latter case (see, e.g., Carlstrom and Fuerst, 2001, or Woodford, 2003), the conditions for real determinacy under a forward looking interest rate rule are, however, more restrictive under non-separable preferences. In the special case where the labor supply elasticity is infinite, $\sigma_l = 0$, the conditions for both cases are identical.²⁷

Now suppose that the central bank applies a constant money growth rule. Then, the model exhibits a relevant state variable, even though the households' behavior is entirely forward

²⁵The condition for real determinacy in part 1 accords to the determinacy conditions in Benhabib et al.'s (2001b) analysis of a continuous time framework, which relates to an end-of-period specification, and in Kurozumi (2004), who applies a Taylor-type interest rate rule featuring current inflation and the output gap.

²⁶Non-emptiness of the sets $(1, \min\{\overline{\rho_{\pi 4}}, \overline{\rho_{\pi 5}}\})$ and $(\max\{1, \overline{\rho_{\pi 4}}\}, \overline{\rho_{\pi 5}})$ requires, $\varepsilon_{ca} < \beta \Upsilon [\chi \sigma_l (z-1)]^{-1}$, and, respectively, $\beta \Upsilon [\chi \sigma_l (z-1)]^{-1} < \varepsilon_{ca} < \Upsilon (1 + \beta) [\chi \sigma_l (z-1)]^{-1}$, if $\overline{\rho_{\pi 4}} < 1$.

²⁷For $\sigma_l = 0$, $\rho_{\pi} \in (1, 1 + 2(1 + \beta)/\chi)$ is necessary and sufficient for local stability and uniqueness of the *E*-version, while it leads to unstable equilibrium sequences in the *B*-version (see proposition 9).

looking. History dependence of the equilibrium sequences is thus induced by monetary policy. It should be noted that this property crucially relies on prices to be sticky. In contrast, a money growth policy cannot not induce a history dependence under perfectly flexible prices (see proposition 7).

Proposition 12 (*E, Money growth policy, $\phi > 0$*) *Suppose that prices are not perfectly flexible, $\phi > 0$, and that end-of-period money enters the utility function. Then, under a constant money growth policy, the equilibrium displays real determinacy, and the equilibrium sequences are locally stable and non-oscillatory.*

Proof. See appendix 6.12.

According to proposition 12, a constant money growth policy guarantees local stability and uniqueness in the *E*-version. In contrast to the *B*-version (see proposition 10), this result is ensured without any further restriction. It should be noted that $\varepsilon_{ca}\phi_{ac} < \sigma_c\sigma_a$, which is ensured by the assumptions (3)-(4), is crucial for the results in proposition 10 and 12. To be more precise, local stability and uniqueness for both timing specifications rely on the assumption that the utility function is strictly concave in consumption and real money balances.

4.3 Discussion

In contrast to the case where prices are flexible, beginning-of-period real balances always serve as a relevant determinant for current consumption in the *B*-version. Nonetheless, the conditions for locally stable, unique, and non-oscillatory equilibrium sequences under interest rate policy unveil the same principles: Whenever real money serves as a relevant predetermined state variable, interest rate policy should be passive to avoid macroeconomic instability (see propositions 8-9). In the *E*-version, however, beginning-of-period real balances are irrelevant and interest rate policy must be active for real determinacy (see proposition 11).

To get an intuition how money demand affects the local dynamics, suppose that the real balance effect is small (low ε_{ca}) and that the economy is hit by a temporary negative productivity shock, $\hat{s}_t < 0$, which induces a rise in current inflation by the aggregate supply relation. For a positive inflation elasticity, $\rho_\pi > 0$, the nominal interest rate rises, causing households to reduce their end-of-period real balances m_t by (22) for the *B*-version as well as by (25) for the *E*-version. According to (21) as well as (24), higher nominal and real interest rates are associated with a rise in the growth rate of consumption and a decline in the growth rate of real balances.²⁸ For the *E*-version, the rise in the nominal interest rate has to induce consumption to grow strong enough such that the growth rate of real balances is also positive and consistent with the consumption Euler equation. Then, consumption \hat{c}_t and real balances \hat{m}_t decline on impact in return to the steady state from below in the subsequent period. Thereby, activeness, $\rho_\pi > 1$, ensures that the real interest rate does not decline, ruling out real indeterminacy, as in a version without real balance effects (see Woodford, 2003).

²⁸According to the real balance effect, a decline in the marginal utility of consumption, can either be brought about by an increase in consumption, or by an decrease in real balances.

If, however, the beginning-of-period stock of money enters the utility function, the rise in the real interest rate is associated with a decline in $(\hat{m}_t - \hat{m}_{t-1})$, causing end-of-period real balances \hat{m}_t to decline, given that the beginning-of-period amount of real balances is predetermined (see 21). Thus, \hat{m}_t declines due to the consumption Euler equation (21) and the money demand condition (22). This, however, further feeds the inflationary pressure, as can be seen from the aggregate supply constraint (20), where real balances enter with a negative coefficient. As a consequence, the contractionary effect of a rise in the nominal and in the real interest rate is perpetuated, giving rise to equilibrium sequences diverging from the steady state. If, however, interest rate policy is passive, real balances tend to grow with the decline in the real interest rate by (21), implying a convergence back to the steady state.

Our results in proposition 10 and 12 indicate that constant money growth ensures locally stable, unique, and non-oscillatory equilibrium sequences for small real balance effects in both versions. To see how this policy rule affects the local dynamics, suppose again that a contractionary productivity shock causes inflation to be higher than in the steady state. According to the constant money growth rule, real balances must decrease. Since real balances and consumption are Edgeworth-complements, households tend to reduce consumption, such that the nominal interest rate and the real interest rate rise due to the consumption Euler equation, (21) or (24). Now recall that the aggregate supply relation (20) or (23) predicts that real balances and consumption affect the current inflation rate in opposite ways: Consumption tends to increase and real money balances tend to decrease current inflation to the extent $\varepsilon_{ca}\chi$. Thus, inflation can only rise if there is a sufficiently large real balance effect, measured by ε_{ca} . Yet, real balance effects are – through our assumptions of a strictly concave utility function (4) – ensured to be sufficiently small such that the decline in consumption indeed governs the inflation response. Thus, the real value of money can subsequently rise due to the decline in inflation, such that the equilibrium sequence of real balances converges to its steady state value.

It should finally be noted that if the utility function is assumed to be separable, our model would reproduce the conventional determinacy results under interest rate rules (see Carlstrom and Fuerst, 2001, or Woodford, 2003): When the central bank follows an interest rate rule featuring current inflation then an active interest rate setting is necessary and sufficient for macroeconomic stability. If the central bank follows a forward looking interest rate rule, activeness is necessary but not sufficient for locally stable and unique equilibrium sequences. To be more precise, $\rho_\pi \in (1, 1 + 2\sigma_c(1 + \beta)[\chi(\sigma_l + \sigma_c)]^{-1})$ ensures local stability and uniqueness. If the monetary authority controls the supply of money then $2z > \sigma_a$ is sufficient for locally stable and unique equilibrium sequences if beginning-of-period money provides utility. Finally, if the stock at the end of the period delivers utility then a constant money growth regime ensures local stability and uniqueness.²⁹ Thus, macroeconomic stability is likely to be ensured by a constant money growth policy, regardless of real balance effects, or if prices are flexible or sticky.

²⁹A similar conclusion is drawn by Evans and Honkapohja (2003) from their numerical analysis of real determinacy under money growth policy in a sticky price model.

5 Conclusion

Real balance effects typically arise when transaction costs are specified in a general equilibrium model in form of shopping time or real resource costs, which are reduced by money holdings. The fact that the equilibrium sequences for real balances and consumption can then not separately be determined, is broadly viewed as negligible for the assessment of monetary policy, given that empirical evidence suggests real balance effects to be relatively small. In contrast to this view, it is demonstrated in this paper that the existence (not the magnitude) of real balance effects can have substantial implications for the determination of a rational expectations equilibrium and of the price level under interest rate policy. However, for real balance effects to contribute to price level determination, as for example suggested by Patinkin (1965), the stock of money held at the beginning of the period rather than held at the end of the period has to be assumed to provide transaction services. Then, there exists a unique initial price level that is consistent with a rational expectations equilibrium, i.e., the equilibrium displays nominal determinacy. Whenever the price level can uniquely be determined, real money serves as a relevant state variable, since money that has been acquired in the previous period relates to the households' current consumption expenditures. These properties, which have until now been ignored in the literature, crucially affect the conditions that ensure macroeconomic stability under interest rate feedback rules regardless whether prices are flexible or sticky. If, on the other hand, current consumption is related to the end-of-period stock of money, then the equilibrium displays nominal indeterminacy, and the well-known principles for uniqueness and stability of equilibrium sequences of a cashless economy (roughly) apply.

While the particular determinacy conditions for macroeconomic stability are highly sensitive under an interest rate policy regime, equilibrium uniqueness and stability is likely to be ensured by a constant money growth policy. This suggests that a central bank that aims to avoid multiple, unstable, or oscillatory equilibrium sequences in an environment where transaction frictions are non-negligible, should rather control the supply of money than the nominal interest rate. Yet, an optimal conduct of monetary policy will certainly require the supply of money to be state contingent (as an interest rate feedback rule), which might be associated with different determinacy implications than a constant money growth regime.

6 Appendix

6.1 Equivalence between explicit transaction frictions and money-in-the-utility-function

In this appendix we examine the relation between the money-in-the-utility-function specification, which is applied throughout the paper, and explicit specifications of transaction frictions, i.e., a shopping time specification and a specification where transactions are associated with real resource costs. For this demonstration, which relates to the analyses in Brock (1974) and Feenstra (1986), we assume, for convenience, that prices are flexible, $\phi = 0$. For both alternative specifications, we assume that the objective of the representative household is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t v(c_t, x_t), \quad v_c > 0, v_{cc} < 0, v_x > 0, v_{cx} = 0, \text{ and } v_{xx} \leq 0, \quad (26)$$

where x denotes leisure.

1. We firstly consider a conventional *shopping time* specification which relates to the one applied in Brock (1974), McCallum and Goodfriend (1987), or Ljungqvist and Sargent (2000). For this we assume that households have to allocate total time endowment, which is normalized to equal one, to leisure x , working time l , and shopping time s , where the shopping time is assumed to depend on the consumption expenditures and on real balances

$$1 \geq x_t + l_t + s_t, \quad \text{where} \quad s_t = H(c_t, A_t/P_t).$$

Following Ljungqvist and Sargent (2000), we assume that the shopping time function H satisfies: $H_c > 0$, $H_{cc} > 0$, $H_a < 0$, and $H_{aa} > 0$ and $H_{ca} \leq 0$. Using that $x_t = 1 - l_t - s_t$ holds in the household's optimum, we can rewrite the utility function as

$$u(c_t, l_t, a_t) = v(c_t, 1 - l_t - H(c_t, a_t)),$$

where $u_c = v_c + v_x(-H_c) \leq 0$, $u_a = v_x(-H_a) > 0$, $u_l = -v_x < 0$, $u_{cc} = v_{cc} + v_{xx}H_c^2 - v_x H_{cc} < 0$, $u_{cl} = v_{xx}H_c \leq 0$, $u_{aa} = v_{xx}H_a^2 - v_x H_{aa} < 0$, $u_{al} = v_{xx}H_a \geq 0$, as well as $u_{ll} = v_{xx}$. Hence, the marginal utility of consumption, which is given by

$$u_{ca} = v_{xx}H_a H_c - v_x H_{ca},$$

is non-decreasing in real balances. If the shopping time function is non-separable or if leisure enters the utility function in a non-linear way, then marginal utility of consumption is strictly increasing in real balances.

2. Next, we closely follow the analysis in Feenstra (1986), and assume that purchases of consumption goods are associated with *real resource costs of transactions* $\phi(c_t, a_t)$, which satisfy: $\phi \geq 0$, $\phi(0, a) = 0$, $\phi_c > 0$, $\phi_a < 0$, $\phi_{cc} \geq 0$, $\phi_{aa} \geq 0$, $\phi_{ac} \leq 0$. The

household's budget constraint then reads

$$M_t + B_t + E_t[q_{t,t+1}Z_{t+1}] + P_t\phi(c_t, a_t) + P_t c_t \leq R_{t-1}B_{t-1} + Z_t + M_{t-1} + P_t w_t l_t + P_t \omega_t - P_t \tau_t. \quad (27)$$

Maximizing (26) subject to (27), a no-Ponzi game condition, and $x_t \leq 1 - l_t$, leads – inter alia – to the following first order conditions for consumption and leisure:

$$\lambda_{rt}(1 + \phi_c(c_t, a_t)) = v_c(c_t), \quad \lambda_{rt}w_t = v_x(1 - l_t),$$

where λ_{rt} denotes the Lagrange multiplier on (27). Note that the aggregate resource constraint now reads $y_t = c_t + \phi(c_t, a_t)$. Using the linear production technology, we therefore obtain the following equilibrium condition: $l_t = c_t + \phi(c_t, a_t)$. Combining these conditions and using that $w_t = (\epsilon - 1)/\epsilon$ under flexible prices, leads to the following expression for the marginal utility of consumption:

$$v_c(c_t) = \frac{\epsilon}{\epsilon - 1} v_x(1 - c_t - \phi(c_t, a_t))(1 + \phi_c(c_t, a_t)).$$

Evidently, the equilibrium sequence of consumption is in general not independent from real money balances due to the existence of transaction costs. Differentiating the latter condition gives

$$\frac{dc_t}{da_t} = \frac{\epsilon/(\epsilon - 1)[v_x\phi_{ca} - v_{xx}(1 + \phi_c)\phi_a]}{v_{cc} + \epsilon/(\epsilon - 1)[(1 + \phi_c)^2 v_{xx} - \phi_{cc}v_x]}.$$

Hence, consumption is positively related to real balances even if either the cross-derivative ϕ_{ca} vanishes or the labor supply elasticity is infinite, i.e. $v_{xx} = 0$.

The corresponding properties of our MIU specification immediately show that an equivalence between the latter and the shopping time specification in 1. requires consumption and real balances to be Edgeworth-complements in the MIU version, if $v_{xx} < 0$ or $H_{ca} < 0$. In order to compare the MIU specification with the transaction cost specification in 2., we apply the first order condition for consumption and labor, the aggregate resource constraint, and the production function, which imply that the equilibrium sequence of consumption under a MIU specification satisfies $dc_t/da_t = -u_{ca}(u_{cc} + \epsilon/(\epsilon - 1)u_{ll})^{-1}$. Evidently, an equivalence between both specifications requires consumption and real balances to be Edgeworth-complements, i.e. $u_{ca} > 0$, if $\phi_{ca} < 0$ or $v_{xx} < 0$. Thus, $v_{xx} < 0$, which implies a finite labor supply elasticity is sufficient for the existence of real balance effects under both specifications of transaction frictions.

6.2 Existence and uniqueness of the steady state

In this appendix, we briefly examine globally the steady state properties of the model. Thereby, we restrict our attention to the case where the nominal interest rate is strictly positive, $R - 1 > 0$. We further omit, for convenience, bars which are throughout the paper used to mark steady state values.

When the stock of money at the beginning of the period enters the utility function, the deterministic steady state is characterized by the following conditions: $-u_l(c) = u_c(c, m/\pi)(\epsilon - 1)/\epsilon$, $R = \pi/\beta$ and $u_a(c, m/\pi)(u_c(c, m/\pi))^{-1} = R - 1$. For an interest rate policy regime, it

is assumed that the policy rule of the central bank, $R(\pi)$, has a unique solution for the steady state relation $R = \pi/\beta$, so that the inflation rate can be substituted out. The first equation implies that c is an implicit function of m , $c = f(m)$, with $f'(m) = -u_{ca}(\epsilon - 1)[R\beta\epsilon(u_{ll} + u_{cc}(\epsilon - 1)/\epsilon)]^{-1} > 0$. Using this, the third equation can be used to determine the steady state value for m with $u_a(f(m), m/\pi)[u_c(f(m), m/(R\beta))]^{-1} = R - 1$. Differentiating the fraction on the left hand side reveals that

$$\frac{du_a/u_c}{dm} = \frac{u_c(\epsilon - 1)/\epsilon(u_{cc}u_{aa} - u_{ca}^2) + u_{ll}(u_{aa}u_c - u_a u_{ca})}{R\beta u_c^2(u_{ll} + u_{cc}(\epsilon - 1)/\epsilon)} < 0,$$

as we assumed concavity for $u(c, a)$. It follows that a globally unique steady state exists if and only if:

$$\lim_{m \rightarrow 0} \frac{u_a(f(m), m/R\beta)}{u_c(f(m), m/(R\beta))} > R - 1.$$

Thus, steady state uniqueness relies on money to be essential (see Obstfeld and Rogoff, 1983): The marginal utility of real money balances should grow with a rate that is higher than the rate by which $1/u_c$ converges to zero when m approaches zero. An analogous line of arguments in case of a money growth policy leads to the condition $\lim_{m \rightarrow 0} u_a(g(m), m/\mu)[u_c(g(m), m/\mu)]^{-1} > \mu/\beta - 1$, where $c = g(m)$ is the implicit relation derived of the steady state condition $-u_l(c) = u_c(c, m/\mu)$ with $g'(m) = -u_{ca}(\epsilon - 1)[\mu\epsilon(u_{ll} + u_{cc}(\epsilon - 1)/\epsilon)]^{-1} > 0$. The condition for existence and uniqueness for the interest rate policy regime if end-of-period money provides transaction services is

$$\lim_{m \rightarrow 0} \frac{u_a(f_E(m), m)}{u_c(f_E(m), m)} > \frac{R - 1}{R},$$

with $f_E(m)' = -u_{ca}(\epsilon - 1)[\epsilon(u_{ll} + u_{cc}(\epsilon - 1)/\epsilon)]^{-1} > 0$. If the monetary authority applies a constant money growth rule then $\lim_{m \rightarrow 0} u_a(f_E(m), m)[u_c(f_E(m), m)]^{-1} > (\mu/\beta - 1)/(\mu/\beta)$ must be satisfied.

6.3 Proof of proposition 2

Consider a monetary policy regime that sets the nominal interest rate contingent on changes in current inflation, $\hat{R}_t = \rho_\pi \hat{\pi}_t$. Reducing the model in (13)-(15) leads to the following system in inflation and real money balances, where we omitted the exogenous state:

$$\begin{pmatrix} E_t \hat{\pi}_{t+1} \\ \hat{m}_t \end{pmatrix} = \begin{pmatrix} \frac{\sigma_l \epsilon_{ca}}{\sigma_l + \sigma_c} + 1 & -\frac{\sigma_l \epsilon_{ca}}{\sigma_l + \sigma_c} \\ \frac{\Upsilon + \sigma_l(\epsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} & -\frac{\Upsilon + \sigma_l(\epsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\sigma_l \epsilon_{ca}}{\sigma_l + \sigma_c} + \rho_\pi - \frac{\sigma_l \epsilon_{ca}}{\sigma_l + \sigma_c} \\ z\rho_\pi & 0 \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix},$$

The characteristic polynomial of A can be simplified to

$$F(X) = X^2 - X\rho_\pi \frac{\Upsilon + \sigma_l(\epsilon_{ca} + \sigma_a) - z\sigma_l \epsilon_{ca}}{\Upsilon + \sigma_l(\epsilon_{ca} + \sigma_a)} - \frac{\rho_\pi z\sigma_l \epsilon_{ca}}{\Upsilon + \sigma_l(\epsilon_{ca} + \sigma_a)}.$$

Consider the case the labor supply elasticity is finite $\sigma_l > 0$. In this case, the determinant of A , $\det(A) = F(0) < 0$, is strictly negative, indicating that exactly one eigenvalue is negative and that real money balances are a relevant state variable. Local stability and uniqueness then requires that there exists exactly one root of $F(X) = 0$ with modulus less than one. To

examine the conditions for this, we use that $F(X)$ further satisfies

$$F(1) = 1 - \rho_\pi,$$

$$F(-1) = \frac{(1 + \rho_\pi)(\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)) - 2z\sigma_l\varepsilon_{ca}\rho_\pi}{\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)}.$$

Thus, for $F(1) < 0$ (> 0) and $F(-1) > 0$ (< 0), the model is locally stable, unique and (non-)oscillatory, since the stable eigenvalue is negative (positive). Suppose that the real balance effect and that the inverse of the labor supply elasticity are large enough such that $\varepsilon_{ca} > \sigma_a(2z - 1)^{-1}$ and $\sigma_l > \bar{\sigma}_l$, where $\bar{\sigma}_l \equiv \frac{\Upsilon}{(2z-1)\varepsilon_{ca}-\sigma_a}$. Then, $F(-1)$ can be negative if ρ_π is sufficiently large. Local stability and uniqueness with $F(1) > 0$ and $F(-1) < 0$, is then ensured by moderate inflation elasticities satisfying $\bar{\rho}_{\pi 1} < \rho_\pi < 1$, where $\bar{\rho}_{\pi 1} \equiv \frac{\sigma_l(\varepsilon_{ca}+\sigma_a)+\Upsilon}{\sigma_l(2z-1)\varepsilon_{ca}-\sigma_l\sigma_a-\Upsilon}$. Alternatively, local stability and uniqueness arise for $F(1) < 0$ and $F(-1) > 0$, which requires $1 < \rho_\pi < \bar{\rho}_{\pi 1}$. Suppose that $\varepsilon_{ca} > \sigma_a(2z - 1)^{-1}$ or $\sigma_l < \bar{\sigma}_l$. Then, $F(-1)$ cannot be negative and local stability and uniqueness then arise if $\rho_\pi > 1$.

Now, consider the case where the labor supply elasticity is infinite, $\sigma_l = 0$. In this case $\det(A) = 0$, indicating that the beginning-of-period value for real money balances is irrelevant for the determination of $\hat{\pi}_t$ and \hat{m}_t . It follows that one eigenvalue equals zero and the other eigenvalue is larger than one, if and only if $\rho_\pi > 1$. Then, the equilibrium sequences for \hat{m}_t , $E_t\hat{c}_{t+1}$, $\hat{\pi}_t$ and \hat{R}_t for $t \geq 0$ are locally stable and uniquely determined, while \hat{c}_t cannot be determined. ■

6.4 Proof of proposition 3

Consider a monetary regime in which future inflation serves as the policy indicator, $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$. Substituting for consumption with (13) and inserting the forward-looking feedback rule, the model in (13)-(15) can be reduced to (where we omitted the exogenous state)

$$\begin{pmatrix} E_t \hat{\pi}_{t+1} \\ \hat{m}_t \end{pmatrix} = \begin{pmatrix} \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} + 1 - \rho_\pi & -\frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} \\ \frac{\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} - z\rho_\pi & -\frac{\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} - \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix}.$$

The characteristic polynomial of \mathbf{A} is given by

$$F(X) = X(X - \frac{\rho_\pi z \sigma_l \varepsilon_{ca}}{(\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a))(1 - \rho_\pi) + \rho_\pi z \sigma_l \varepsilon_{ca}}).$$

Evidently, real money balances are not a relevant state variable, and one can only solve for \hat{m}_t , $E_t \hat{\pi}_{t+1}$, $E_t \hat{c}_{t+1}$ and $\hat{R}_t \forall t \geq 0$. For a finite labor supply elasticity, $\sigma_l > 0$, local stability and uniqueness requires the other eigenvalue (one is equal to zero) to be unstable. A positive unstable root arises if monetary policy is active and $\sigma_l > \bar{\sigma}_{l2}$ or if $1 < \rho_\pi < \bar{\rho}_{\pi 2}$ for $\sigma_l < \bar{\sigma}_{l2}$ or $\varepsilon_{ca} < \sigma_a/(2z - 1)$. A negative unstable root exists if $\rho_\pi > \bar{\rho}_{\pi 2}$, given that $\sigma_l > \bar{\sigma}_l$ and $\varepsilon_{ca} > \sigma_a/(2z - 1)$, for $\sigma_l < \bar{\sigma}_{l2}$ or $\varepsilon_{ca} < \sigma_a/(2z - 1)$. Thus, $1 < \bar{\rho}_{\pi 2} < \rho_\pi < -\bar{\rho}_{\pi 1}$ leads to a locally stable and unique equilibrium with a negative root for $\sigma_l < \bar{\sigma}_l$ or $\varepsilon_{ca} < \sigma_a/(2z - 1)$. When the labor supply elasticity is infinite, $\sigma_l = 0$, then the Euler equation reads $(1 - \rho_\pi)E_t \hat{\pi}_{t+1} = 0$. Thus, the model displays local stability and uniqueness if and only if $\rho_\pi \neq 1$. ■

6.5 Proof of proposition 4

Under a constant money growth regime the nominal interest rate can be substituted out so that the reduced form system of the model in (13)-(15) reads (where we omitted the exogenous state)

$$\varpi_1 E_t[\hat{c}_{t+1}] - (\varpi_2 + 1) \hat{m}_t + \varpi_2 E_t[\hat{\pi}_{t+1}] = -\sigma_c \hat{c}_t + \varepsilon_{ca} \hat{m}_{t-1} - \varepsilon_{ca} \hat{\pi}_t, \quad (28)$$

$$\varepsilon_{ca} \hat{m}_{t-1} = (\sigma_l + \sigma_c) \hat{c}_t + \varepsilon_{ca} \hat{\pi}_t, \quad (29)$$

where $\varpi_1 \equiv (\sigma_c(1 - z) + \phi_{ac})z^{-1}$ and $\varpi_2 \equiv (\varepsilon_{ca}(1 - z) - z + \sigma_a)z^{-1}$, and $\hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t$. After eliminating consumption with (29) and inflation with the linearized money growth rule (16), we get the following difference equation in \hat{m}_t :

$$E_t \hat{m}_{t+1} = \frac{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c)}{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c) - (\Upsilon + \sigma_l \varepsilon_{ca} + \sigma_l \sigma_a)} \hat{m}_t.$$

Once \hat{m}_t is determined, which requires an unstable root, one can solve for $\hat{\pi}_t$ and $\hat{c}_t \forall t \geq 1$, while the initial values for consumption \hat{c}_0 and inflation $\hat{\pi}_0$ cannot be determined. Local uniqueness and stability of the equilibrium sequences $\{\hat{m}_t, \hat{\pi}_{t+1}, \hat{c}_{t+1}, \hat{R}_t\}_{t=0}^\infty$ thus require $\left| \frac{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c)}{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c) - (\Upsilon + \sigma_l \varepsilon_{ca} + \sigma_l \sigma_a)} \right| > 1$. If $z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c) - (\Upsilon + \sigma_l \varepsilon_{ca} + \sigma_l \sigma_a) > 0$, then the root is positive and unstable. Rearranging and using $\Upsilon = \sigma_c \sigma_a - \varepsilon_{ca} \phi_{ac}$ shows that this conditions is satisfied for $z > \sigma_a$. ■

6.6 Proof of proposition 6

Consider the case where the central bank sets the nominal interest rate contingent on changes in current inflation, $\hat{R}_t = \rho_\pi \hat{\pi}_t$. After substituting for consumption and eliminating \hat{m}_t and \hat{m}_{t+1} with the static money demand equation (19), one obtains the following difference equation (where we omitted the exogenous state):

$$(d + 1)\rho_\pi \hat{\pi}_t = (d\rho_\pi + 1)E_t \hat{\pi}_{t+1},$$

where $d \equiv (z - 1)\sigma_l \varepsilon_{ca}[\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)]^{-1} > 0$. Therefore $\rho_\pi > 1$ is necessary and sufficient for local stability and uniqueness of the equilibrium sequences of inflation $\hat{\pi}_t$, real balances \hat{m}_t , consumption \hat{c}_t and the nominal interest rate, $\hat{R}_t \forall t \geq 0$.

Now, consider the case where future inflation serves as the policy indicator, $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$. When the labor supply elasticity is finite, $\sigma_l > 0$, then the model in (17)-(19) reduces to:

$$E_t \hat{\pi}_{t+2} = \frac{\rho_\pi(1 + d) - 1}{d\rho_\pi} E_t \hat{\pi}_{t+1}.$$

Evidently, one cannot determine current inflation rate $\hat{\pi}_t$. One obtains a unique and locally stable solution for expected inflation, and the current values of consumption, real money balances and the nominal interest rate, if the eigenvalue of this equation is positive and unstable, which requires $\rho_\pi > 1$. Alternatively, $\rho_\pi < [1 + 2d]^{-1}$ ensures local stability and uniqueness, where one eigenvalue is smaller than -1 . When the labor supply elasticity is infinite, $\sigma_l = 0$, then uniqueness of a equilibrium sequence for $E_t \hat{\pi}_{t+1} \forall t \geq 0$ is guaranteed by $\rho_\pi \neq 1$. ■

6.7 Proof of proposition 7

Under a constant money growth policy, $\hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t$, the model in (17)-(19) can – by eliminating the nominal interest rate – be reduced to (where we omitted the exogenous state):

$$\varepsilon_{ca}\hat{m}_t = (\sigma_l + \sigma_c)\hat{c}_t, \quad (30)$$

$$\gamma_1 E_t \hat{c}_{t+1} + \gamma_2 E_t \hat{\pi}_{t+1} + \gamma_3 \hat{m}_t = \left(\gamma_1 + \frac{\sigma_c + \phi_{ac}}{z}\right)\hat{c}_t, \quad (31)$$

where $\gamma_1 = \sigma_c(z-1)z^{-1} > 0$, $\gamma_2 = (1 + \varepsilon_{ca})(z-1)z^{-1} > 0$ and $\gamma_3 = (\varepsilon_{ca} + \sigma_a)z^{-1} > 0$. Eliminating consumption with (30) and inflation with the linearized money growth rule leads to the following difference equation in real money balances:

$$E_t \hat{m}_{t+1} = \frac{[\sigma_l(1 + \varepsilon_{ca}) + \sigma_c](z-1) + \Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)}{[\sigma_l(1 + \varepsilon_{ca}) + \sigma_c](z-1)} \hat{m}_t,$$

which evidently exhibits an unstable root. Thus, one can uniquely determine end-of-period real balances \hat{m}_t , current consumption \hat{c}_t , the nominal interest rate $\hat{R}_t \forall t \geq 0$, while inflation $\hat{\pi}_t$ can only be determined for $t \geq 1$. ■

6.8 Proof of proposition 8

When current inflation enters the reaction function of the central bank, $\hat{R}_t = \rho_\pi \hat{\pi}_t$, then the model in (20)-(22) can be rewritten as (where we omitted the exogenous state)

$$\begin{pmatrix} \hat{m}_t \\ E_t \hat{\pi}_{t+1} \\ E_t \hat{c}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{m}_{t-1} \\ \hat{\pi}_t \\ \hat{c}_t \end{pmatrix},$$

where $\mathbf{A} = \begin{pmatrix} 0 & \beta & 0 \\ -\varepsilon_{ca} - \sigma_a & \varepsilon_{ca} + \sigma_a & \sigma_c + \phi_{ac} \\ -\varepsilon_{ca} & \varepsilon_{ca} + 1 & \sigma_c \end{pmatrix}^{-1} \begin{pmatrix} \chi \varepsilon_{ca} & 1 - \chi \varepsilon_{ca} - \chi(\sigma_l + \sigma_c) \\ 0 & z \rho_\pi & 0 \\ -\varepsilon_{ca} & \rho_\pi + \varepsilon_{ca} & \sigma_c \end{pmatrix}.$

The characteristic polynomial of the matrix A reads

$$\begin{aligned} F(X) &= X^3 - X^2 \frac{\Upsilon(1 + \beta + \chi) + \sigma_l \chi(\varepsilon_{ca} + \sigma_a)}{\beta \Upsilon} \\ &+ X \frac{\Upsilon(1 + \chi \rho_\pi) + \chi \sigma_l \rho_\pi(\varepsilon_{ca} + \sigma_a) - z \sigma_l \rho_\pi \chi \varepsilon_{ca}}{\beta \Upsilon} + \frac{z \sigma_l \rho_\pi \chi \varepsilon_{ca}}{\beta \Upsilon}. \end{aligned}$$

The value of $F(X)$ at $X = 0$, which is given by $F(0) = \frac{z \sigma_l \rho_\pi \chi \varepsilon_{ca}}{\beta \Upsilon} \geq 0$, discloses that $F(0) > 0$ if $\sigma_l > 0$ and $F(0) = 0$ if $\sigma_l = 0$. Hence, for $\sigma_l > 0$ the model either exhibits one or three negative roots, as $\det(\mathbf{A}) = -F(0) < 0$. Continuing the analysis for this case ($\sigma_l > 0$), the value of $F(X)$ at $X = 1$, given by

$$F(1) = \frac{(\rho_\pi - 1)\chi(\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a))}{\beta \Upsilon},$$

reveals that locally stable, unique, and non-oscillatory equilibrium sequences necessarily require $\rho_\pi < 1$. In order to rule out any further stable root for this case, we examine $F(X)$ at

$$X = -1,$$

$$F(-1) = \frac{\sigma_l (2z\chi\rho_\pi\varepsilon_{ca} - \chi\varepsilon_{ca} - \chi\rho_\pi\sigma_a - \chi\rho_\pi\varepsilon_{ca} - \chi\sigma_a) - \Upsilon(2\beta + \chi + \chi\rho_\pi + 2)}{\Upsilon\beta},$$

Suppose that $(2z - 1)\varepsilon_{ca} > \sigma_a$. Then, $F(-1) < 0$ requires, $\sigma_l > \bar{\sigma}_l = \frac{\Upsilon}{(2z-1)\varepsilon_{ca}-\sigma_a}$, and $\rho_\pi > \bar{\rho}_{\pi 3} \equiv \frac{\sigma_l\chi(\varepsilon_{ca}+\sigma_a)+\Upsilon(2\beta+\chi+2)}{\chi[\sigma_l(2z-1)\varepsilon_{ca}-\sigma_l\sigma_a-\Upsilon]}$, which implies that the negative root is smaller than minus one. If $\rho_\pi > \bar{\rho}_{\pi 3}$, and $\sigma_l > \bar{\sigma}_l$, we know that $F(-1) > 0$, such that the two remaining roots are unstable and the equilibrium is locally stable and unique. Since the unstable roots are of opposite sign, there cannot exist a pair of complex roots in this case.

If $\rho_\pi > 1$, local stability and uniqueness require $F(-1)$ to be negative, implying that there is a stable root between minus one and zero, i.e. an oscillatory equilibrium path. Then, local stability and uniqueness are ensured by $\rho_\pi > \max\{1, \beta\Upsilon(z\sigma_l\chi\varepsilon_{ca})^{-1}\}$ if $\sigma_l < \bar{\sigma}_l$. If $\sigma_l > \bar{\sigma}_l$ then $\max\{1, \beta\Upsilon(z\sigma_l\chi\varepsilon_{ca})^{-1}\} < \rho_\pi < \bar{\rho}_{\pi 3}$ ensures local stability and uniqueness. As $\rho_\pi > \frac{\beta\Upsilon}{z\sigma_l\chi\varepsilon_{ca}} \Rightarrow \det(\mathbf{A}) < -1$, a pair of stable roots (either complex or real) cannot exist, since they would necessarily lead to a determinant with an absolute value that is smaller than one. Thus, there exists exactly one stable eigenvalue. Until now we assumed that $(2z - 1)\varepsilon_{ca} > \sigma_a$. If $0 < (2z - 1)\varepsilon_{ca} < \sigma_a$ then no non-oscillatory equilibrium exists, and the equilibrium is stable and uniquely determined if $\rho_\pi > \max\{1, \beta\Upsilon(z\sigma_l\chi\varepsilon_{ca})^{-1}\}$.

If $\sigma_l = 0$, the characteristic polynomial reduces to $F(X)|_{\sigma_l=0} = (1 - X + X^2\beta - X\chi - X\beta + \chi\rho_\pi)\frac{X}{\beta}$. Hence, one root equals zero and the remaining roots are given by $X^2 - \frac{\chi+\beta+1}{\beta}X + \frac{1+\chi\rho_\pi}{\beta} = 0$. As $F(0) = \frac{1+\chi\rho_\pi}{\beta} > 0$, $F'(1) = -\frac{1-\beta+\chi}{\beta} < 0$, and $F(1) = \chi\frac{\rho_\pi-1}{\beta}$, activeness, $\rho_\pi > 1$, is then necessary and sufficient for local stability and uniqueness. ■

6.9 Proof of proposition 9

Applying the linearized forward looking interest rate rule, $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$ and eliminating consumption, the model in (20)-(22) reads (where we omitted the exogenous state)

$$\phi_2 \hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \phi_1 \hat{m}_{t-1}, \quad (32)$$

$$\phi_3 \hat{m}_t - \phi_5 E_t \hat{\pi}_{t+1} = \phi_3 \hat{m}_{t-1} - \phi_4 \hat{\pi}_t, \quad (33)$$

where $\phi_1 \equiv \chi\kappa^{-1}[\Upsilon + \sigma_l(\varepsilon_{ca} + \sigma_a)] > 0$, $\phi_2 \equiv 1 + \phi_1 - \chi\kappa^{-1}(\sigma_l + \sigma_c)z\rho_\pi$, $\phi_3 \equiv \Upsilon\kappa^{-1} > 0$, $\phi_4 \equiv \phi_3 - \sigma_c\kappa^{-1}z\rho_\pi$, $\phi_5 \equiv \phi_4 - (1 - \rho_\pi)$, $\kappa \equiv \sigma_c + \phi_{ac} > 0$, and $\Upsilon \equiv \sigma_c\sigma_a - \varepsilon_{ca}\phi_{ac} > 0$. Rewriting the model (32)-(33) gives

$$\begin{pmatrix} \hat{m}_t \\ E_t \hat{\pi}_{t+1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\beta}\frac{\phi_1}{\phi_3}\phi_5 + 1 - \frac{1}{\phi_3}\phi_4 + \frac{1}{\beta}\frac{\phi_2}{\phi_3}\phi_5 & \\ -\frac{1}{\beta}\phi_1 & \frac{1}{\beta}\phi_2 \end{pmatrix} \begin{pmatrix} \hat{m}_{t-1} \\ \hat{\pi}_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{m}_{t-1} \\ \hat{\pi}_t \end{pmatrix},$$

The characteristic polynomial of \mathbf{A} is given by

$$F(X) = X^2 - \frac{(1 + \beta + \chi - \chi\rho_\pi)\Upsilon + \chi\sigma_l(\sigma_a + \varepsilon_{ca} - \rho_\pi(\sigma_a + \varepsilon_{ca} - z\varepsilon_{ca}))}{\beta\Upsilon}X + \frac{\Upsilon + z\chi\rho_\pi\sigma_l\varepsilon_{ca}}{\beta\Upsilon}.$$

Given that $\det(\mathbf{A}) = F(0) = \beta^{-1}\Upsilon^{-1}(\Upsilon + z\chi\rho_\pi\sigma_l\varepsilon_{ca}) > 1$, two stable roots and thus indeterminacy can be ruled out. The value of $F(X)$ at $X = 1$, which is given by $F(1) =$

$\chi \frac{\Upsilon + \sigma_a \sigma_l + \sigma_l \varepsilon_{ca}}{\beta \Upsilon} (\rho_\pi - 1)$, further reveals that $\rho_\pi < 1$ is necessary and sufficient for $F(1) < 0$ and thus for the existence of exactly one stable root, which lies between zero and one. Local stability and uniqueness can, however, also be ensured by an active policy $\rho_\pi > 1$ if $F(X)$ at $X = -1$, which is given by

$$F(-1) = \frac{1}{\beta \Upsilon} [\Upsilon (2 + 2\beta + \chi) + \chi \sigma_l (\sigma_a + \varepsilon_{ca}) + \rho_\pi \chi (\sigma_l ((2z - 1) \varepsilon_{ca} - \sigma_a) - \Upsilon)],$$

is negative, such that there is exactly one stable root between minus one and zero. This, necessarily requires $\sigma_l ((2z - 1) \varepsilon_{ca} - \sigma_a) < \Upsilon$ and $\rho_\pi > \frac{\chi \sigma_l (\sigma_a + \varepsilon_{ca}) + \Upsilon (2 + 2\beta + \chi)}{\chi (\sigma_l (1 - 2z) \varepsilon_{ca} + \sigma_l \sigma_a + \Upsilon)} = -\overline{\rho_\pi 3}$, and leads to oscillatory equilibrium sequences. ■

6.10 Proof of proposition 10

Eliminating the nominal interest rate and applying the linearized money growth rule (16), the model in (20)-(22) reads (where we omitted the exogenous state)

$$\varpi_1 E_t \hat{c}_{t+1} - [\varpi_2 + 1] \hat{m}_t + \varpi_2 E_t \hat{\pi}_{t+1} = -\sigma_c \hat{c}_t + \varepsilon_{ca} \hat{m}_{t-1} - \varepsilon_{ca} \hat{\pi}_t, \quad (34)$$

$$\hat{\pi}_t = \chi [(\sigma_l + \sigma_c) \hat{c}_t - \varepsilon_{ca} \hat{m}_{t-1} + \varepsilon_{ca} \hat{\pi}_t] + \beta E_t \hat{\pi}_{t+1}, \quad (35)$$

$$\hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t, \quad (36)$$

where $\varpi_1 \equiv (\sigma_c(1 - z) + \phi_{ac}) z^{-1}$ and $\varpi_2 \equiv (\varepsilon_{ca}(1 - z) - z + \sigma_a) z^{-1}$. The model in (34)-(36) can be rewritten as:

$$\begin{pmatrix} E_t \hat{c}_{t+1} \\ E_t \hat{\pi}_{t+1} \\ \hat{m}_t \end{pmatrix} = \begin{pmatrix} \varpi_1 & \varpi_2 - \varpi_2 - 1 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\sigma_c & -\varepsilon_{ca} & \varepsilon_{ca} \\ -\chi(\sigma_l + \sigma_c) & 1 - \chi \varepsilon_{ca} & \chi \varepsilon_{ca} \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{c}_t \\ \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix}.$$

The characteristic polynomial of \mathbf{A} can be reduced to:

$$F(X) = X^3 - X^2 \frac{\varpi_1(1 + \beta - \chi \varepsilon_{ca}) - \beta \sigma_c + \varpi_2 \chi (\sigma_c + \sigma_l)}{\varpi_1 \beta} - X \frac{\sigma_c(\beta + 1 + \chi) - \varpi_1 + \chi \sigma_l (\varepsilon_{ca} + 1)}{\varpi_1 \beta} + \frac{\sigma_c}{\varpi_1 \beta}.$$

In order to disclose the conditions for local stability and uniqueness, we examine $F(X)$ at $X = 0$ and $X = 1$. Using that $F(0)$ is given by $F(0) = -z \sigma_c \beta^{-1} ((z - 1) \sigma_c - \phi_{ac})^{-1}$, we can conclude that $F(0) < -1 \Leftrightarrow \phi_{ac} < (z - 1) \sigma_c$. The latter condition further ensures $F(X)$ at $X = 1$, which is given by

$$F(1) = \chi \frac{\sigma_l (\sigma_a + \varepsilon_{ac}) + \Upsilon}{\beta ((z - 1) \sigma_c - \phi_{ac})},$$

to be positive, $F(1) > 0 \Leftrightarrow \phi_{ac} < (z - 1) \sigma_c$. In this case, there exists at least one stable root, $X_1 \in (0, 1)$, and, as $\det(\mathbf{A}) = -F(0) > 1$, at least one of the remaining two roots must be unstable. Thus, there cannot exist another (real or complex) stable eigenvalue (see proof of proposition 8). In order to ensure that both remaining roots are unstable, $F(X)$ has to be

negative at $X = -1$, provided that $\phi_{ac} < (z - 1) \sigma_c$. Since $F(-1)$ is given by

$$F(-1) = -\frac{1}{\beta((z-1)\sigma_c - \phi_{ac})} \left[\begin{aligned} &(2z - \sigma_a) \chi (\sigma_c + \sigma_l) + (z - 1) (2\sigma_c + 2\beta\sigma_c + \chi\sigma_l\varepsilon_{ca}) \\ &+ (z\sigma_c - \phi_{ac}) (2 + 2\beta) + \chi\varepsilon_{ca} (\phi_{ac} + z\sigma_l) \end{aligned} \right],$$

we can conclude that $F(-1) < 0$ and that the model is locally stable and unique, given that $\sigma_a < 2z$ and $\phi_{ac} < (z - 1) \sigma_c$. ■

6.11 Proof of proposition 11

We start with the case where the nominal interest rate is set contingent on current inflation, $\hat{R}_t = \rho_\pi \hat{\pi}_t$. Reducing the model (23)-(25) into a two-dimensional system in inflation and consumption gives (where we omitted the exogenous state)

$$\begin{pmatrix} E_t \hat{\pi}_{t+1} \\ E_t \hat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 1 + \varpi_b \rho_\pi \sigma_c - \varpi_a \end{pmatrix}^{-1} \begin{pmatrix} (1 - \chi \varpi_b \rho_\pi) - \chi (\sigma_l + \sigma_c) + \chi \varpi_a \\ (1 + \varpi_b) \rho_\pi & \sigma_c - \varpi_a \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \end{pmatrix},$$

where $\varpi_a = \varepsilon_{ca} \frac{\phi_{ac} + \sigma_c}{\varepsilon_{ca} + \sigma_a} > 0$, $\varpi_b = \varepsilon_{ca} \frac{z-1}{\varepsilon_{ca} + \sigma_a} > 0$. The characteristic polynomial of the matrix \mathbf{A} , can be simplified to

$$F(X) = X^2 - X \frac{\Upsilon(1 + \beta + \chi) + \chi\sigma_l(\sigma_a + \varepsilon_{ca} + (z-1)\rho_\pi\varepsilon_{ca})}{\beta\Upsilon} + \frac{\Upsilon(1 + \chi\rho_\pi) + \chi\rho_\pi\sigma_l(\sigma_a + z\varepsilon_{ca})}{\beta\Upsilon}.$$

with $\det(\mathbf{A}) = F(0) = \frac{\Upsilon(1 + \chi\rho_\pi) + \chi\rho_\pi\sigma_l(\sigma_a + z\varepsilon_{ca})}{\beta\Upsilon} > 1$. Hence, local stability and uniqueness are ensured if $F(X)$ is positive at $X = 1, -1$. While $F(X)$ is, regardless of interest rate policy, strictly positive at $X = -1$,

$$F(-1) = \frac{\Upsilon(2 + 2\beta + \chi + \chi\rho_\pi) + \chi\sigma_l(\sigma_a + \varepsilon_{ca} + \rho_\pi\sigma_l + (2z - 1)\rho_\pi\varepsilon_{ca})}{\beta\Upsilon} > 0,$$

the value for $F(X)$ at $X = 1$, which is given by $F(1) = (\rho_\pi - 1)\chi \frac{\Upsilon + \sigma_l\sigma_a + \sigma_l\varepsilon_{ca}}{\beta\Upsilon}$, is positive if interest rate policy satisfies $\rho_\pi > 1$. Hence, the equilibrium sequences are for $\sigma_l \geq 0$ locally stable and unique if and only if $\rho_\pi > 1$.

Now, consider the case, where future inflation serves as the indicator for the policy rule, $R_t = \rho_\pi E_t \pi_{t+1}$. After eliminating inflation with the money demand condition (25), the model can be reduced to the following 2×2 system in real balances and consumption (where we omitted the exogenous state)

$$\begin{pmatrix} E_t \hat{m}_{t+1} \\ E_t \hat{c}_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{m}_t \\ \hat{c}_t \end{pmatrix},$$

$$\text{where } \mathbf{A} = \begin{pmatrix} -\chi\varepsilon_{ca} - \beta\eta_a \chi (\sigma_l + \sigma_c) + \beta\eta_b \\ -\varepsilon_{ca} & \sigma_c \end{pmatrix}^{-1} \begin{pmatrix} -\eta_a & \eta_b \\ -(\rho_\pi - 1)\eta_a - \varepsilon_{ca} (\rho_\pi - 1)\eta_b + \sigma_c \end{pmatrix}.$$

where $\eta_a = \frac{\varepsilon_{ca} + \sigma_a}{(z-1)\rho_\pi} > 0$ and $\eta_b = \frac{\phi_{ac} + \sigma_c}{(z-1)\rho_\pi} > 0$. The characteristic polynomial of \mathbf{A} is given

by:

$$F(X) = X^2 + \frac{\Upsilon}{\beta\Upsilon + (1-z)\sigma_l\rho_\pi\varepsilon_{ca}\chi} - X \frac{\Upsilon(1+\beta+\chi(1-\rho_\pi)) + \chi\sigma_l(\sigma_a + \varepsilon_{ca} - \sigma_a\rho_\pi - z\rho_\pi\varepsilon_{ca})}{\beta\Upsilon + (1-z)\sigma_l\rho_\pi\varepsilon_{ca}\chi},$$

where $\det(\mathbf{A}) = F(0) = \Upsilon(\beta\Upsilon - (z-1)\chi\rho_\pi\sigma_l\varepsilon_{ca})^{-1}$, implying that $F(0) > 1$ and that there is at least one unstable eigenvalue, if $\rho_\pi < \overline{\rho_{\pi 4}}$, where $\overline{\rho_{\pi 4}} \equiv \frac{\beta\Upsilon}{\chi\sigma_l\varepsilon_{ca}(z-1)}$. Hence, the equilibrium sequences are locally stable and unique (two unstable roots) when $F(X)$ is positive at $X = 1, -1$, provided that $\rho_\pi < \overline{\rho_{\pi 4}}$. The value of $F(X)$ at $X = 1$, which is given by

$$F(1) = \frac{(\Upsilon + \sigma_a\sigma_l + \sigma_l\varepsilon_{ca})(\rho_\pi - 1)\chi}{\beta\Upsilon + (1-z)\sigma_l\rho_\pi\varepsilon_{ca}\chi},$$

reveals that $F(1) > 0$ if $1 < \rho_\pi < \overline{\rho_{\pi 4}}$, which rules out the existence of a positive stable root. To establish that there is no negative stable root, the value for $F(X)$ at $X = -1$

$$F(-1) = \frac{\Upsilon(2+2\beta+\chi) + \chi\sigma_l(\sigma_a + \varepsilon_{ca}) - \rho_\pi(\chi\sigma_l(2z\varepsilon_{ca} + \sigma_a - \varepsilon_{ca}) + \chi\Upsilon)}{\beta\Upsilon + (1-z)\sigma_l\rho_\pi\varepsilon_{ca}\chi},$$

has to be positive, $F(-1) > 0$, which is ensured by $\rho_\pi < \overline{\rho_{\pi 5}}$, where $\overline{\rho_{\pi 5}} \equiv \frac{\Upsilon(2+2\beta+\chi) + \chi\sigma_l(\sigma_a + \varepsilon_{ca})}{\chi\sigma_l(2z\varepsilon_{ca} + \sigma_a - \varepsilon_{ca}) + \chi\Upsilon}$. Hence, for $\sigma_l > 0$ the model is locally stable and unique if $1 < \rho_\pi < \min\{\overline{\rho_{\pi 4}}, \overline{\rho_{\pi 5}}\}$. Local stability and uniqueness, however, is also ensured if $F(X)$ is negative at $X = -1, 0$, and 1 , which requires $\max\{1, \overline{\rho_{\pi 4}}\} < \rho_\pi < \overline{\rho_{\pi 5}}$. For $\sigma_l = 0$, the condition reads $\rho_\pi \in (1, 1 + 2(1+\beta)/\chi)$. ■

6.12 Proof of proposition 12

Eliminating the nominal interest rate and applying the linearized money growth rule (16), the model in (23)-(25) reads

$$-\chi\varepsilon_{ca}\widehat{m}_t + \beta E_t\widehat{\pi}_{t+1} = -\chi(\sigma_l + \sigma_c)\widehat{c}_t + \widehat{\pi}_t + \chi\sigma_l\widehat{s}_t, \quad (37)$$

$$\gamma_1 E_t\widehat{c}_{t+1} + \gamma_2 E_t\widehat{\pi}_{t+1} + \gamma_3\widehat{m}_t = (\gamma_1 + \frac{\sigma_c + \phi_{ac}}{z})\widehat{c}_t, \quad (38)$$

$$\widehat{m}_t = \widehat{m}_{t-1} - \widehat{\pi}_t, \quad (39)$$

where $\gamma_1 = \sigma_c(z-1)z^{-1} > 0$, $\gamma_2 = (1+\varepsilon_{ca})(z-1)z^{-1} > 0$ and $\gamma_3 = (\varepsilon_{ca} + \sigma_a)z^{-1} > 0$. Then the model in (37)-(39) can be rewritten as (where we omitted the exogenous state):

$$\begin{pmatrix} \widehat{m}_t \\ E_t\widehat{\pi}_{t+1} \\ E_t\widehat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} -\chi\varepsilon_{ca} & \beta & 0 \\ \gamma_3 & \gamma_2 & \gamma_1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & -\chi(\sigma_l + \sigma_c) \\ 0 & 0 & \gamma_1 + \frac{\sigma_c + \phi_{ac}}{z} \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{m}_{t-1} \\ \widehat{\pi}_t \\ \widehat{c}_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \widehat{m}_{t-1} \\ \widehat{\pi}_t \\ \widehat{c}_t \end{pmatrix}.$$

The characteristic polynomial of the matrix \mathbf{A} is given by

$$F(X) = X^3 - X^2 \frac{\gamma_1 z (2\beta + 1 - \chi \varepsilon_{ca}) + \beta(\sigma_c + \phi_{ac}) + \chi \gamma_2 z (\sigma_l + \sigma_c)}{\gamma_1 z \beta} \\ + X \frac{\gamma_1 z (2 + \beta - \chi \varepsilon_{ca}) + (1 + \beta - \chi \varepsilon_{ca})(\phi_{ac} + \sigma_c) + (\sigma_l + \sigma_c)(\gamma_2 + \gamma_3) z \chi}{\gamma_1 z \beta} - \frac{\gamma_1 z + \sigma_c + \phi_{ac}}{\gamma_1 z \beta}.$$

To derive the conditions for local stability and uniqueness, which requires one stable and two unstable roots, we examine $F(X)$ at $X = 0$, $F(0) = -\frac{1}{\beta}(1 + \frac{1}{\sigma_c} \frac{\phi_{ac} + \sigma_c}{z-1}) < -1$. As $F(X)$ at $X = 1$, which is given by

$$F(1) = \chi[\sigma_l(\varepsilon_{ca} + \sigma_a) + \varepsilon_{ca} \Upsilon] (\gamma_1 z \beta)^{-1} > 0,$$

is strictly positive, there exists one real stable eigenvalue, lying between zero and one. Hence, there must be at least one unstable eigenvalue, given that $\det(\mathbf{A}) = -F(0) > 1$. Thus, the two remaining (complex or real) roots cannot both be stable. Given that $F(X)$ is strictly negative at $X = -1$,

$$F(-1) = -\frac{(4\gamma_1 z + 2(\phi_{ac} + \sigma_c))(1 + \beta) + \chi[\Upsilon + 2z\gamma_1 + \gamma_2 z(\sigma_l + \sigma_c) + \sigma_l(2z\gamma_2 + z\gamma_3)]}{\gamma_1 z \beta} < 0,$$

the remaining root cannot lie between minus one and zero, indicating that the model is locally stable and unique. ■

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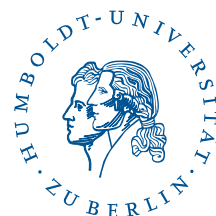
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